5.7 Multiple Integrals

We now restrict our attention to scalar valued functions. In order to simplify the notation and make drawing pictures easier, we'll also assume the domain is two-dimensional. However,

Fix a rectangle

$$R = [a,b] \times [c,d] \subseteq \mathbb{R}^2$$

For partitions P of $[a,b]$ and Q of $[c,d]$:

$$P = \{a = x_0 < x_1 < \cdots < x_m = b\} \quad Q = \{c = y_0 < y_1 \cdots < y_n = d\}$$

we let $G = P \times Q$ and consider

$$R_{ij} := I_i \times I_j := [x_{i-1}, x_i] \times [y_{j-1}, y_j] \qquad \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix}$$

Denote

$$\Delta x_i = x_i - x_{i-1}$$
$$\Delta y_j = y_j - y_{j-1}$$

and let

$$|R_{ij}| = \Delta x_i \Delta y_j$$

be the area of R_{ij}. Finally, we choose a set S sample points:

$$S = \{(s_{ij}, t_{ij}) \in R_{ij} : 1 \le i \le m, 1 \le j \le n\}$$

Def: For $f: R \to \mathbb{R}$, the Riemann sum of f corresponding to the grid $G = P \times Q$ and sample points S is the quantity:

$$R(f, G, S) := \sum_{i=1}^{m} \sum_{j=1}^{n} f(s_{ij}, t_{ij}) \cdot |R_{ij}|$$

The mesh of G is the quantity

$$\mathrm{mesh}(G) = \max_{(i,j)} \sqrt{\Delta x_i^2 + \Delta y_j^2} \quad (= \mathrm{diam}(R_{ij}))$$

Def: For $f: R \to \mathbb{R}$, we say f is Riemann integrable on R if there exists a number $A \in \mathbb{R}$ s.t. $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. if $\mathrm{mesh}(G) < \delta$ then for any S

$$|A - R(f, G, S)| < \varepsilon.$$

That is,
$$\lim_{\mathrm{mesh}(G)\to 0} R(f,G,S) = A.$$
We denote $\int_R f := A$.

Def: Given bounded $f: R \to \mathbb{R}$ and a grid G, define
$$m_{ij} = \inf\{f(x,y) : (x,y)\in R_{ij}\}$$
$$M_{ij} = \sup\{f(x,y) : (x,y)\in R_{ij}\}.$$
The lower and upper sums of f with respect to G are the quantities:
$$L(f,G) = \sum_{i=1}^{m}\sum_{j=1}^{n} m_{ij}\,|R_{ij}|$$
$$U(f,G) = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij}\,|R_{ij}|,$$
respectively. By monotonicity, the following limits always exist:
$$\underline{\int_R} f = \lim_{\mathrm{mesh}(G)\to 0} L(f,G)$$
$$\overline{\int_R} f = \lim_{\mathrm{mesh}(G)\to 0} U(f,G)$$
technically requires nets, but mono. is clear via common refinements

and it follows that
$$\underline{\int_R} f \le \overline{\int_R} f.$$

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The following results hold via ~~similar~~ the same arguments used in the one-dim'l case:

Prop: Let $f: R \to \mathbb{R}$.

(a) If f is Riemann integrable on R, then it is bdd.

(b) The set of — " — functions, $\mathcal{L}^1(R)$, is a vector space and integration is a linear transformation from $\mathcal{L}^1(R)$ to $\mathbb{R}$.

(c) $\int_R k = k\,|R|$ for k a constant.

(d) If $g: R \to \mathbb{R}$ is s.t. $f \le g$ and both R. int'ble
$$\int_R f \le \int_R g$$

(e) If f is bdd, $\underline{\int}_R f = \overline{\int}_R f$ iff $\int_R f$ exists, in which case all three are equal.

Riemann – Lebesgue Theorem

We want to prove an upgraded version of the fact that a function discontinuous at a finite number of points is Riemann integrable.

Def: A subset $Z \subseteq \mathbb{R}^2$ is called a zero set (or a null set, or "has Lebesgue measure zero") if $\forall \varepsilon > 0$ $\exists$ a countable collection of open rectangles $\{S_\lambda\}_{\lambda \in \Lambda}$ s.t.

$$Z \subseteq \bigcup_{\lambda \in \Lambda} S_\lambda \quad \text{but} \quad \sum_{\lambda \in \Lambda} |S_\lambda| < \varepsilon.$$

~~Ex: $Z = \mathbb{Q} \times \mathbb{Q}$ is a zero set.~~
~~Let $\mathbb{Q} = (q_n)_{n\in\mathbb{N}}$ be an enumeration~~

Ex ① $Z = \mathbb{R} \times \{0\}$ is a zero set. Given ε,

let $S_n = (n-1, n+1) \times \left(\frac{-\delta}{2^{|n|}}, \frac{\delta}{2^{|n|}}\right)$

Then

$$\sum_{n\in\mathbb{Z}} |S_n| = \sum_{n\in\mathbb{Z}} 2 \cdot \frac{2\delta}{2^{|n|}} = \delta \sum_{n\in\mathbb{Z}} \frac{1}{2^{|n|-2}} \le \delta \sum_{n=0}^{\infty} \frac{1}{2^{n-3}} = 16\delta.$$

② Exercise Show: any finite set is a zero set

skip? ③ $Z = \mathbb{Q} \times \mathbb{Q}$ is a zero set. Let $\mathbb{Q} = (q_n)_{n\in\mathbb{N}}$ and set

$$S_{m,n} = \left(q_n - \frac{\delta}{2^{n+1}}, q_n + \frac{\delta}{2^{n+1}}\right) \times \left(q_m - \frac{\delta}{2^{m+1}}, q_m + \frac{\delta}{2^{m+1}}\right)$$

So

$$\sum_{m,n\in\mathbb{N}} |S_{m,n}| = \sum \frac{2\delta}{2^{n+1}} \cdot \frac{2\delta}{2^{m+1}} = \delta^2 \left(\sum_n \frac{1}{2^n}\right)\left(\sum_m \frac{1}{2^m}\right) = \delta^2$$

"Use $\varepsilon = \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$ to"

Exercise: Show any countable union of zero sets is still a zero set.

<u>Def</u> For $f: \mathbb{R} \to \mathbb{R}$ bounded, the <u>oscillation</u> at $z \in \mathbb{R}$ is the quantity
$$\operatorname{osc}_z(f) := \lim_{r \to 0} \{ \sup(f(B(z,r))) - \inf(f(B(z,r))) \}$$
ball centered at z with radius r

<u>Remark</u>: Note that this limit always exists since
$$\sup(f(B(z,r))) \searrow \text{ as } r \searrow 0$$
$$\inf(f(B(z,r))) \nearrow \text{ as } r \searrow 0$$

<u>Exercise</u>: Show that f is continuous at $z \in \mathbb{R}$ iff $\operatorname{osc}_z(f) = 0$.

<u>Def</u>: Fix $f: \mathbb{R} \to \mathbb{R}$. For each $k \in \mathbb{N}$, define
$$D_k = \{ z \in \mathbb{R} : \operatorname{osc}_z(f) \geq 1/k \}$$
Then the <u>discontinuity set</u> of f is the union
$$D = \bigcup_{k \in \mathbb{N}} D_k.$$

<u>Thm (Riemann – Lebesgue Theorem)</u>
For $f: \mathbb{R} \to \mathbb{R}$ bounded, it is Riemann integrable if and only if its discontinuity set is a zero set.

<u>Proof</u>: ($\Longrightarrow$) Assume f is Riemann integrable. Fix $k \in \mathbb{N}$. Then
$$\underline{\int}_R f = \overline{\int}_R f$$
so given $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever G is a grid with $\operatorname{mesh}(G) < \delta$, then
$$* \quad U(f,G) - L(f,G) < \varepsilon.$$
Fix such a grid G. If R_{ij} contains some $z \in D_k$ in its interior, then
$$M_{ij} - m_{ij} \geq 1/k$$
The total area of all such R_{ij} is at most $k\varepsilon$, because of $(*)$. All other points in D_k lie on the lines of the grid: $x_i \times [c,d]$ and $[a,b] \times y_j$, but these lines are a zero set. Since $\varepsilon > 0$ was arbitrary, $k\varepsilon$ can be made arbitrarily

small for fixed k. and hence D_k is a zero set. Applying this to each $k \in \mathbb{N}$, we see that D is a zero set as the countable union of zero sets.

($\Longleftarrow$) Assume D is a zero set. Fix $k \in \mathbb{N}$, then D_k is also a zero set. By def. of D_k, every $z \in R \setminus D_k$ has an open nbhd W_z s.t.

$$\sup\{f(w) : w \in W_z\} - \inf\{f(w) : w \in W_z\} < 1/k.$$

Now, because D_k is a zero set, we can cover it with open rectangles with small total area, say

$$\sum |S_\ell| < \sigma.$$

Let $\mathcal{V}$ be the open cover of R consisting of the S_ℓ's and the W_z's. Since R is compact, there is a positive Lebesgue number $\lambda > 0$ associated to $\mathcal{V}$; that is, for any subset $S \subseteq R$ with $\mathrm{diam}(S) < \lambda$, S is completely contained in some S_ℓ or W_z. Take G to be a grid with $\mathrm{mesh}(G) < \lambda$. Then $\mathrm{diam}(R_{ij}) < \lambda$ for each i, j. Consider

$$U(f,G) - L(f,G) = \sum (M_{ij} - m_{ij}) \cdot |R_{ij}|$$

$$= \sum_{R_{ij} \subseteq W_z} (M_{ij} - m_{ij}) |R_{ij}| + \sum_{R_{ij} \subseteq S_\ell} (M_{ij} - m_{ij}) |R_{ij}|$$

In the first sum, $M_{ij} - m_{ij} < 1/k$, so the total sum is less than $|R|/k$. In the second sum, $\sum |R_{ij}| \leq \sum |S_\ell| < \sigma$. So if $M = \sup\{|f(z)| : z \in R\}$ then the second sum is less than $2M \cdot \sigma$. Hence

$$U(f,G) - L(f,G) < \frac{|R|}{k} + 2M\sigma$$

So by choosing large k and small σ, we have shown the upper and lower sums are arbitrarily close. Hence $\underline{\int}_R f = \overline{\int}_R f$ and f is Riemann integrable. $\square$

Remark: The Riemann–Lebesgue theorem also holds for ~~f~~ bounded $f:[a,b]\to\mathbb{R}$ (and you may have proven it in Math 104) where a zero set in $\mathbb{R}$ is the same as a zero set in $\mathbb{R}^2$ but you cover by open intervals and sum their lengths.

• We can now consider what new ideas the multivariable setting offers: order of integration

Def: For $f:R\to\mathbb{R}$ bounded, the lower and upper slice integrals are:

$$\underline{F}(y) := \underline{\int_a^b} f(x,y)\,dx \quad\text{and}\quad \overline{F}(y) = \overline{\int_a^b} f(x,y)\,dx$$

respectively, for $y\in[c,d]$.

Thm (Fubini's Theorem) If f is Riemann integrable, then so are $\underline{F}$ and $\overline{F}$. Moreover,

$$\int_R f = \int_c^d \underline{F}\,dy = \int_c^d \overline{F}\,dy.$$

Pf: Let $G = P\times Q$ be a grid formed by partitions P of $[a,b]$ and Q of $[c,d]$. We claim

$$L(f,G) \le L(\underline{F},Q) \left(= \sum_{j=1}^{n} \inf\{\underline{F}(y): y_{j-1}\le y\le y_j\}\cdot \Delta y_j\right)$$

Indeed, fix a partition subinterval $J_j \subseteq [c,d]$. Then if $y\in J_j$,

$$m_{ij} = \inf\{f(p): p\in R_{ij}\} \le \inf\{f(s,y): s\in I_i\} =: m_i(f(\cdot,y))$$

Thus, for all $y \in J_j$, we have

$$\sum_{i=1}^{m} m_{ij} \Delta x_i \le \sum_{i=1}^{m} m_i(f(\cdot, y)) \Delta x_i = L(f(\cdot, y), P) \le \underline{F}(y).$$

Letting $m_j(\underline{F}) := \inf\{\underline{F}(y) : y \in J_j\}$, we have

$$\sum_{i=1}^{m} m_{ij} \Delta x_i \le m_j(\underline{F}).$$

Therefore

$$L(f, G) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij} \Delta x_i \Delta y_j \le \sum_{j=1}^{n} m_j(\underline{F}) \Delta y_j = L(\underline{F}, Q)$$

A similar argument yields $U(F, G) \le U(f, G)$.
Thus

$$L(f, G) \le L(\underline{F}, Q) \le \begin{matrix} U(\underline{F}, Q) \\ L(\overline{F}, Q) \end{matrix} \le U(\overline{F}, Q) \le U(f, G).$$

Taking sup's over G on the left and inf's on the right yields:

$$\int_R f \le \sup_Q L(\underline{F}, Q) = \underline{\int_c^d} \underline{F}(y)\,dy \le \overline{\int_c^d} \underline{F}(y)\,dy = \inf_Q U(\underline{F}, Q) \le \int_R f$$

Since the upper and lower integrals of $\underline{F}$ agree, it is Riemann integrable with

$$\int_c^d \underline{F}(y)\,dy = \int_R f.$$

Similarly for $\overline{F}$. $\square$

<u>Remark</u>: Since $\underline{F} \le \overline{F}$, the equality of their integrals in Fubini's theorem implies

$$\{y \in [c,d] : \underline{F}(y) < \overline{F}(y)\}$$

is a zero set. Hence for each y <u>not</u> in this set,

$$\int_a^b f(x, y)\,dx$$

exists. For y in the set, the integral need not exist, but since it happens "rarely" we still write:

$$\int_R f = \iint_R f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Cor If $f: R \to \mathbb{R}$ is Riemann integrable, then
$$\int_R f = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Pf: Apply Fubini's theorem to x instead of y. $\square$

There is a geometric consequence of Fubini's theorem concerning computing areas. For a set $S \subseteq \mathbb{R}^2$, recall that
$$\partial S = \overline{S} \setminus S^\circ$$
$$= \{ x \in \mathbb{R}^2 : \; B(x,r) \cap S \neq \emptyset \text{ and } B(x,r) \cap S^c \neq \emptyset \;\; \forall r > 0 \}$$

Cor (Cavalieri's Principle) Let $S \subseteq R \subseteq \mathbb{R}^2$. If ∂S is a zero set, then
$$\text{area}(S) = \int_a^b \text{length}(S_x)\,dx$$
where $S_x = S \cap \{x\} \times \mathbb{R}$.

Pf: Consider $f: R \to \mathbb{R}$
$$f(x,y) = \chi_S(x,y) = \begin{cases} 1 & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$
(the characteristic function of S). The discontinuity set of f is precisely ∂S (exercise). Thus the Riemann-Lebesgue theorem implies it is Riemann integrable, and clearly
$$\text{area}(S) = \int_R \chi_S$$
Since $f(x,y) = \chi_{S_x}(y)$, we have by Fubini's theorem:
$$\text{area}(S) = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_a^b \text{length}(S_x)\,dx. \quad \square$$

Change of Variables Formula

Let $U, W \subseteq \mathbb{R}^2$ be open sets, let $\varphi: U \to W$ be a C^1-diffeomorphism, let $R \in U$ be a rectangle and let $f: W \to \mathbb{R}$ be Riemann integrable.

<u>Def</u> The <u>Jacobian</u> of φ at $z \in U$ is the quantity
$$\mathrm{Jac}_z \varphi := \det\left((D\varphi)_z \right)$$

Our goal is to prove the following formula:
$$\int_R f \circ \varphi \cdot |\mathrm{Jac}\, \varphi| = \int_{\varphi(R)} f$$
While the left-hand-side makes sense, we don't yet know how to integrate over regions besides rectangles, regions like $\varphi(R)$. So we need to develop the theory a bit more.

<u>Def</u> Given a bounded subset $S \subseteq \mathbb{R}^2$, we define its <u>characteristic function</u> $\chi_S : \mathbb{R}^2 \to \mathbb{R}$ by
$$\chi_S(p) = \begin{cases} 1 & \text{if } p \in S \\ 0 & \text{otherwise.} \end{cases}$$
If χ_S is Riemann integrable over a/any rectangle containing S, we say S is <u>Riemann measurable</u> and define its <u>area</u> (or <u>Jordan content</u>) by
$$\mathrm{area}(S) := \int \chi_S.$$

<u>Remark</u>: We have already ~~observed~~ observed that S is Riemann measurable if and only if ∂S is a zero set.

- For $S = R$ a rectangle, clearly $\mathrm{area}(R) = |R|$. Thus for ~~general~~ $S \subseteq \mathbb{R}^2$ Riemann measurable we write
$$|S| := \mathrm{area}(S) = \int \chi_S.$$

We now prove a special case of the change of variables formula:

<u>Prop</u> If $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^2)$ is an isomorphism, then for every Riemann measurable set $S \subseteq \mathbb{R}^2$, $T(S)$ is Riemann measurable with

$$|T(S)| = |\det T| \cdot |S|$$

<u>Pf</u>: Let A be the matrix rep of T (w.r.t. the std basis for $\mathbb{R}^2$). From linear algebra, we know

$$A = E_1 \cdots E_k$$

where $E_1, \ldots, E_k$ are elementary 2×2 matrices of the form

$$\begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 & \sigma \\ 0 & 1 \end{bmatrix} \qquad \begin{matrix} \lambda > 0 \\ \sigma \in \mathbb{R} \end{matrix}$$

Since

$$\det(T) = \det(E_1) \cdots \det(E_k)$$

it suffices to prove the proposition for T an elementary matrix.

Let $I^2 = [0,1] \times [0,1]$ in $\mathbb{R}^2$. Then the first three types of elementary matrices transform I^2 into

$$[0,\lambda] \times [0,1], \quad [0,1] \times [0,\lambda], \text{ and } I^2$$

respectively. In particular, we have

$$|E(I^2)| \overset{!}{=} |\det(E)| \cdot |I^2|$$

each time. For the fourth type of elem. matrix, I^2 is transformed into the parallelogram

$$\Pi := \{(x,y) \in \mathbb{R}^2 : \sigma y \leq x \leq 1 + \sigma y \text{ and } 0 \leq y \leq 1\}.$$

This is Riemann measurable because we can easily see that the boundary set (four line segments) is a zero set. So by Fubini's Theorem

$$|\Pi| = \int \chi_\Pi = \int_0^1 \left[\int_{\sigma y}^{1+\sigma y} 1 \, dx\right] dy = \int_0^1 1 \, dy = 1 = |\det E| \cdot |I^2|.$$

Thus the Prop holds for I^2. The same methods show for any rectangle R, that

$$* \quad |E(R)| = |\det(E)| \cdot |R|.$$

We claim that this implies the formula for any Riemann measurable S.

Let $\varepsilon > 0$, and choose a grid G on some $R \supset S$ with mesh(G) so small that the subrectangles $R_{ij} \subseteq R$ of G satisfy:

$$** \quad |S| - \varepsilon \le \sum_{R_{ij} \subseteq S} |R_{ij}| \le \sum_{R_{ij} \cap S \ne \emptyset} |R_{ij}| \le |S| + \varepsilon.$$

This is possible using the Riemann integrability of S; the first sum is $L(\chi_S, G)$ and the second is $U(\chi_S, G)$.

Note that if R_{ij}° denotes the interior of R_{ij} (i.e. if $R_{ij} = [a,b] \times [c,d]$, $R_{ij}^\circ = (a,b) \times (c,d)$), then these are all disjoint and hence $\forall z \in \mathbb{R}^2$

$$\sum_{R_{ij} \subset S} \chi_{R_{ij}^\circ}(z) \le \chi_S(z)$$

Since an elementary matrix E is invertible, $E(R_{ij}^\circ) = E(R_{ij})^\circ$, which are also all disjoint so

$$\sum_{R_{ij} \subset S} \chi_{E(R_{ij})^\circ}(z) \le \chi_{E(S)}(z) \qquad z \in \mathbb{R}^2.$$

Observe that since $\partial(E(R_{ij})^\circ)$ is a zero set,

$$|E(R_{ij})| = |E(R_{ij})^\circ| = \int \chi_{E(R_{ij})^\circ}.$$

Thus, by linearity and monotonicity of the integral

$$\sum_{R_{ij} \subset S} |E(R_{ij})| = \sum_{R_{ij} \subset S} \int \chi_{E(R_{ij})^\circ} = \int \sum_{R_{ij} \subseteq S} \chi_{E(R_{ij})^\circ} = \underline{\int} \sum_{R_{ij}} \chi_{E(R_{ij})^\circ}$$

$$\le \underline{\int} \chi_{E(S)}.$$

Similarly, $\chi_{E(S)}(z) \le \sum_{R_{ij} \cap S \ne \emptyset} \chi_{E(R_{ij})}(z)$, so that

$$\overline{\int} \chi_{E(S)} \le \int \sum_{R_{ij} \cap S \ne \emptyset} \chi_{E(R_{ij})} = \sum_{R_{ij} \cap S \ne \emptyset} |E(R_{ij})|$$

Combining these estimates with (*) and (**) we get

$$|\det(E)| \cdot (|S| - \varepsilon) \le |\det(E)| \sum_{R_{ij} \subseteq S} |R_{ij}| = \sum_{R_{ij} \subseteq S} |E(R_{ij})|$$

$$\le \underline{\int} \chi_{E(S)} \le \overline{\int} \chi_{E(S)} \le \sum_{R_{ij} \cap S \ne \emptyset} |E(R_{ij})| \le |\det(E)| (|S| + \varepsilon)$$

Letting $\varepsilon \to 0$ we see

$$|\det(E)| \cdot |S| = \underline{\int} \chi_{E(S)} = \overline{\int}_{\mathbb{R}} \chi_{E(S)}$$

Thus $E(S)$ is Riemann measurable with $|E(S)| = |\det(E)| \cdot |S|$ $\square$

<u>Remark</u>: This prop. is exactly the change of variables formula for $\varphi = T$, $R = S$, and $f = 1$. It holds in higher dim. and is called the <u>volume multiplier formula</u>. In fact, this can used to define $\det(T)$ for $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

2/23/2018

Before proving the change of variables formula, we require two lemmas:

<u>Lemma 1</u>: Let $0 \in U \subseteq \mathbb{R}^2$ be open. Suppose $\psi : U \to \mathbb{R}^2$ is C^1, $\psi(0) = 0$, and that

$$\varepsilon := \sup_{u \in U} \| D\psi|_u - Id \| < \infty.$$

If $r > 0$ is s.t. $B(0,r) \subseteq U$, then

$$\psi(B(0,r)) \subseteq B(0, (1+\varepsilon) \cdot r).$$

<u>Pf</u>: By the C^1-MVT, for $u \in U$ s.t. $[0,u] \subseteq U$

$$\psi(u) = \psi(u) - \psi(0) = \int_0^1 (D\psi)_{tu} \, dt \, (u)$$

$$= \int_0^1 [(D\psi)_{tu} - Id] \, dt \, (u) + u$$

If $|u| < r$, then

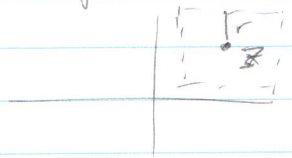
$$|\psi(u)| \leq \int_0^1 \|(D\psi)_{tu} - Id\| \, dt \cdot r + r$$

$$\leq \varepsilon r + r = (1+\varepsilon) \cdot r. \quad \square$$

<u>Remark</u>: Note that for $|\cdot|$, we only used that $|\cdot|$ satisfied the triangle inequality. Hence Lemma 1 is valid for any norm on $\mathbb{R}^2$. In particular for:

$$|(x,y)|_\infty = \max\{|x|, |y|\}.$$

Recall that for this norm



$B(z,r)$.

<u>Lemma 2</u>: Let $Z \subseteq \mathbb{R}^2$ be a zero set. Suppose $h: Z \to \mathbb{R}^2$ is Lipschitz:

$$L := \sup\left\{ \frac{|h(x)-h(y)|}{|x-y|} : x \neq y \text{ in } Z \right\} < \infty.$$

Then $h(Z) \subseteq \mathbb{R}^2$ is a zero set.

<u>Pf</u>: Let $\varepsilon > 0$. We can find a countable covering of Z by open <u>squares</u> S_k s.t.

$$\sum_k |S_k| < \varepsilon.$$

(<u>Exercise</u>: Show we can choose squares instead of rectangles)

Note that for all $x, y \in S_k \cap Z$

$$|h(x) - h(y)| \le L \cdot |x-y| \le L \cdot \mathrm{diam}(S_k).$$

Hence

$$\mathrm{diam}(h(S_k \cap Z)) \le L\, \mathrm{diam}(S_k)$$

Thus we can find a square S_k' of ~~edge~~ side length $L \cdot \mathrm{diam}(S_k)$ that contains $h(S_k \cap Z)$. The squares S_k' cover $h(Z)$ and

$$\sum |S_k'| \le L^2 \sum_k (\mathrm{diam}(S_k))^2 = 2L^2 \sum_k |S_k| \le 2L^2 \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $h(Z)$ is a zero set. $\square$

<u>Thm</u> (Change of Variables Formula)

Let $U, W \subseteq \mathbb{R}^2$ be open sets and let $\varphi: U \to W$ be a C^1-diffeomorphism. For $R \subseteq U$ a rectangle and $f: W \to \mathbb{R}$ Riemann integrable we have

$$\int_R f \circ \varphi \, |\mathrm{Jac}(\varphi)| = \int_{\varphi(R)} f$$

<u>Proof</u>: We first must argue that each side makes sense. If D' is the set of discontinuity points of $f|_{\varphi(R)}$ then $D := \varphi^{-1}(D')$ are the

discontinuity points of $f \circ \varphi$ in R. In fact, since $|\mathrm{Jac}(\varphi)|$ is cts (by virtue of φ being C^1) D is the set of discontinuity points of the $f \circ \varphi \, |\mathrm{Jac}\,\varphi|$. So if D is a zero set, the R+L thm implies it is Riemann integrable.

Note that $\varphi(R)$ is a compact set, and in particular is bounded. Hence the C^1-MVT implies φ^{-1} is Lipschitz on $\varphi(R)$. So by Lemma 2, D is a zero set. Thus the left-hand side makes sense.

For the right-hand side, ∂R is a zero set and

$$\partial \varphi(R) = \varphi(\partial R)$$

so by Lemma 2, this is a zero set and $\chi_{\varphi(R)}$ is Riemann integrable. Let $R' \supseteq \varphi(R)$ be a rectangle. Then what we mean by the right-hand side is:

$$\int_{\varphi(R)} f = \int_{R'} f \cdot \chi_{\varphi(R)}.$$

which is defined. It remains to establish the equality.

Equip $\mathbb{R}^2$ with the max norm:

$$|(x,y)|_\infty = \max\{|x|, |y|\}.$$

Consider $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ with the induced operator norm:

$$\|T\|_\infty = \sup\left\{ \frac{|Tv|_\infty}{|v|_\infty} : v \in \mathbb{R}^2 \setminus \{0\} \right\}$$

Let $\varepsilon > 0$. Let $r > 0$, which we will determine later. Take G a grid on R with $\mathrm{mesh}(G) < r$. Let z_{ij} be the center of R_{ij} and denote:

$$A_{ij} = (D\varphi)_{z_{ij}} \qquad \varphi(z_{ij}) = w_{ij} \qquad W_{ij} = \varphi(R_{ij})$$

The Talyor approx to φ at z_{ij} is then:

$$\phi_{ij}(z) := w_{ij} + A_{ij}(z - z_{ij})$$

~~Consider~~ Since A_{ij} is invertible, ϕ_{ij} is invertible.
Consider $\psi_{ij} = \phi_{ij}^{-1} \circ \varphi$. This sends z_{ij} to itself and

$$(D\psi_{ij})_{z_{ij}} = (D\phi_{ij}^{-1})_{w_{ij}} \cdot (D\varphi)_{z_{ij}} = A_{ij}^{-1} \cdot A_{ij} = Id.$$

In general

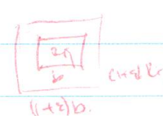
$$(D\psi_{ij})_z = A_{ij}^{-1} \cdot (D\varphi)_z.$$

Now, since R is compact, $D\varphi$ is unif. cts on it. Thus we can choose $r>0$ small enough so for all $z \in R_{ij}$ and all ij we have

$$\| (D\psi_{ij})_z - id \| < \varepsilon.$$

Applying Lemma 1 to ~~$\phi_{ij}(z) - z_{ij}$~~ $\psi_{ij}(z+z_{ij}) - z_{ij}$ $0 \mapsto 0$
which has same deriv as ψ_{ij} yields

$$\phi_{ij}^{-1} \circ \varphi(R_{ij}) \subseteq (1+\varepsilon)R_{ij}$$



↑ $(1+\varepsilon)$-dilation of R_{ij} centered at z_{ij}.

The same argument applied to $\varphi^{-1} \circ \phi_{ij}$ with radius $r/(1+\varepsilon)$ instead of r yields

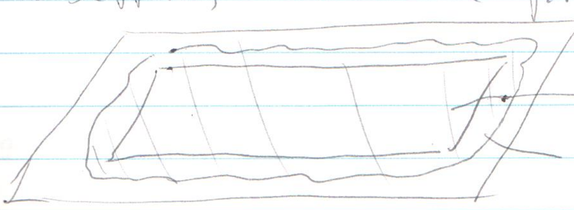
$$\varphi^{-1} \circ \phi_{ij}\left(\tfrac{1}{(1+\varepsilon)} R_{ij}\right) \subset R_{ij}$$

Together, these imply

$$\phi_{ij}((1+\varepsilon)^{-1} R_{ij}) \subset \varphi(R_{ij}) = W_{ij} \subset \phi_{ij}((1+\varepsilon)R_{ij})$$

still a rectangle because using l^∞

Since ϕ_{ij} is affine, ↑ and ⌒ are parallelograms:



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By the volume multiplier formula we obtain:

$$|\phi_{ij}((1+\varepsilon)^{-1}R_{ij})| = |\det(A_{ij})| \cdot |(1+\varepsilon)^{-1}R_{ij}|$$
$$= \frac{|\mathrm{Jac}_{z_{ij}}\varphi| \cdot |R_{ij}|}{(1+\varepsilon)^2}$$
$$\le |W_{ij}|$$
$$\le (1+\varepsilon)^2 \, |\mathrm{Jac}_{z_{ij}}\varphi| \, |R_{ij}|.$$

Equivalently, setting $J_{ij} := |\mathrm{Jac}_{z_{ij}}\varphi|$, we have

$$\frac{1}{(1+\varepsilon)^2} \le \frac{|W_{ij}|}{J_{ij}\,|R_{ij}|} \le (1+\varepsilon)^2$$

$\xRightarrow{\text{Exercise}}$
$$\big|\, |W_{ij}| - J_{ij}\,|R_{ij}| \,\big| \le 4\varepsilon \cdot J_{ij}\,|R_{ij}|$$
$$\le 4\varepsilon\, J\, |R_{ij}| \qquad *$$

where $J := \sup\{|\mathrm{Jac}_z\varphi| : z \in R\}$.

Now, let m_{ij} and M_{ij} be the inf and sup, respectively, of $f\circ\varphi$ on R_{ij}. Then for all $w \in \varphi(R)$ we have

$$\sum m_{ij}\, \chi_{W_{ij}^\circ}(w) \le f(w) \le \sum M_{ij}\, \chi_{W_{ij}}(w).$$

Integrating this inequality over $\varphi(R)$ yields:

$$\sum m_{ij}|W_{ij}| \le \int_{\varphi(R)} f(w) \le \sum M_{ij}\,|W_{ij}|$$

Using ($*$) we obtain

$$\sum m_{ij}\, J_{ij}\,|R_{ij}| - 4\varepsilon \cdot M \cdot J\,|R| \le \int_{\varphi(R)} f \le \sum M_{ij}\, J_{ij}\,|R_{ij}| + 4\varepsilon M J |R|$$

where $M = \sup\{|f\circ\varphi(z)| : z \in R\}$. There are ~~upper~~ lower and ~~lower~~ upper sums for $f\circ\varphi \cdot |\mathrm{Jac}_z\varphi|$, so we obtain

$$\int_R f\circ\varphi \cdot |\mathrm{Jac}_z\varphi| - 4\varepsilon M J|R| \le \int_{\varphi(R)} f \le \int_R f\circ\varphi\,|\mathrm{Jac}_z\varphi| + 4\varepsilon M J |R|.$$

Letting $\varepsilon \to 0$ yields the formula. $\square$

Cor Let $U, W \subseteq \mathbb{R}^2$ be open sets and let $\varphi: U \to W$ be a C^1-diffeomorphism. For $S \subseteq U$ Riemann measurable and $f: W \to \mathbb{R}$ Riemann integrable

$$\int_S f\circ\varphi\,|\mathrm{Jac}_z(\varphi)| = \int_{\varphi(S)} f$$

<u>Pf</u>: Let $R \supset S$ be a rectangle. Then we apply the change of variables formula to $g = f\chi_{\varphi(S)}$.

$$\int_{\varphi(S)} f = \int_{\varphi(R)} g = \int_R g\circ\varphi\,|\mathrm{Jac}_z(\varphi)|$$

Note that

$$g\circ\varphi(z) = \begin{cases} f\circ\varphi(z) & \text{if } z\in S \\ 0 & \text{if } z\notin S \end{cases}$$

$$= f\circ\varphi\cdot\chi_S$$

So

$$\int_{\varphi(S)} f = \int_R f\circ\varphi\cdot\chi_S\,|\mathrm{Jac}_z\varphi| = \int_S f\circ\varphi\,|\mathrm{Jac}\,\varphi|.$$

Exercise

Of course, the validity of these integrals follows from the fact that

$$\partial\varphi(S) = \varphi(\partial S)$$

is a zero set since φ is Lipschitz on R and, the discontinuity set of g is contained in the union of the discontinuity set of f and $\partial\varphi(S)$, while the discontinuity set of $f\circ\varphi\cdot|\mathrm{Jac}_z\varphi|$ is contained in the union of $\varphi^{-1}(\{\text{disc. set of } f\})$ and ∂S. □

<u>Integrating in higher dimensions</u>

For $n\geq 2$, consider a <u>box</u>

$$R = [a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]\subseteq\mathbb{R}^n$$

Let $f: R\to\mathbb{R}$, we form Riemann sums similar to before: let $P_1,\dots,P_n$ be partitions of $[a_1,b_1],\dots,[a_n,b_n]$, respectively. Then

$$G = P_1\times\cdots\times P_n$$

is a grid, which divides R into sub-boxes R_ℓ. We take a sample point $s_\ell\in R_\ell$ for each ℓ and let $S = \{s_\ell\}$.

Then

$$R(f, G, S) = \sum_{\ell} f(s_\ell) \cdot |R_\ell|$$

is a Riemann sum for f, where $|R_\ell|$ is the product of ~~the~~ its edge lengths (which we think of as its volume).

We define Riemann integrability in the same way as in $n=2$ case, and all the properties of the Riemann integral hold here as well.

In particular, 1 the Riemann + Lebesgue Thm holds, where a zero set $Z \subseteq \mathbb{R}^n$ is s.t. $\forall \varepsilon > 0$ Z can be covered by countably many open boxes R_ℓ satisfying

$$\sum_\ell |R_\ell| < \varepsilon.$$

2 Fubini's theorem also holds: by induction:

$$\int_R f = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} f(x_1, \ldots, x_n)\, dx_n \cdots dx_1$$

and this integral can be computed in any order

3 We can also prove the Change of Variables formula here, ~~that (the Volume~~ where $\mathrm{Jac}_z(\varphi)$ is the determinate of $(D\varphi)_z \in M(n, n)$. Moreover, the volume multiplier formula also holds, we just need to consider more elementary matrices.

5.8 Differential Forms

Recall Stokes theorem from Calculus: $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$

$\vec{F}$ let $S \subseteq \mathbb{R}^3$ be a smooth surface with simple closed bounding curve C. Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_S \mathrm{curl}(\vec{F}) \cdot d\vec{S}.$$

You also learned Green's theorem and the divergence