Exercises:

1. Consider the two following bases for $\mathbb{P}_2$:

$$\mathcal{S} := \{1, x, x^2\} \qquad \text{and} \qquad \mathcal{A} = \{x-1, x^2+x, 2x\}.$$

   (a) Compute $[I]_{\mathcal{A}}^{\mathcal{S}}$, the change of coordinate matrix from $\mathcal{A}$ to $\mathcal{S}$.

   (b) Compute $[I]_{\mathcal{S}}^{\mathcal{A}}$, the change of coordinate matrix from $\mathcal{S}$ to $\mathcal{A}$.

   (c) Compute the following coordinate vectors:
   - $[3x^2 - x + 2]_{\mathcal{A}}$
   - $[x^2 + x - 3]_{\mathcal{A}}$
   - $[x^2 + x]_{\mathcal{A}}$

   (d) For $T\colon \mathbb{P}_2 \to \mathbb{P}_2$ defined by $T(p(x)) = p'(x)$, compute the following matrix representations:
   - $[T]_{\mathcal{S}}^{\mathcal{S}}$
   - $[T]_{\mathcal{A}}^{\mathcal{S}}$
   - $[T]_{\mathcal{S}}^{\mathcal{A}}$
   - $[T]_{\mathcal{A}}^{\mathcal{A}}$

2. Define a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ by letting $T(\mathbf{v})$ be the reflection of $\mathbf{v}$ over the line $y = -\frac{1}{3}x$. For the standard basis $\mathcal{S} = \{\mathbf{e}_1, \mathbf{e}_2\}$, compute $[T]_{\mathcal{S}}^{\mathcal{S}}$.

3. Show that if $A, B \in M_{n\times n}$ are similar, then $\text{Tr}(A) = \text{Tr}(B)$.

4. Consider the matrix
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and let $\mathbf{v}_1, \mathbf{v}_2$ be its column vectors. Prove that the area of the parallelogram determined by $\mathbf{v}_1, \mathbf{v}_2$ is always $|ad - bc|$. [**Hint:** find a rotation matrix R_θ such that $R_\theta \mathbf{v}_1 = \alpha \mathbf{e}_1$ for some scalar α.]

5. Let $A \in M_{n\times m}$.

   (a) Suppose $n = m$. Prove that $A^T A$ is invertible if and only if AA^T is invertible.

   (b) Suppose $n \neq m$. Find a counterexample to the previous statement.

6. Fix $m, n \in \mathbb{N}$.

   (a) Show that $\det \begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix} = \det(E)$ for an elementary matrix $E \in M_{m\times m}$.

   (b) Show that $\det \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & E \end{pmatrix} = \det(E)$ for an elementary matrix $E \in M_{n\times n}$.

   (c) Show that $\det \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} = \det(A)\det(C)$ for $A \in M_{m\times m}$, $B \in M_{m\times n}$, and $C \in M_{n\times n}$.

   [**Hint:** use a product and the first two parts.]

Solutions:

1. (a) $[I]_{\mathcal{A}}^{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}.$

   (b) $[I]_{\mathcal{S}}^{\mathcal{A}} = ([I]_{\mathcal{A}}^{\mathcal{S}})^{-1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$

(c)
- $[3x^2 - x + 2]_{\mathcal{A}} = [I]_{\mathcal{S}}^{\mathcal{A}}[3x^2 - x + 2]_{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ -1 \end{pmatrix}$
- $[x^2 + x - 3]_{\mathcal{A}} = [I]_{\mathcal{S}}^{\mathcal{A}}[3x^2 + x - 3]_{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -3/2 \end{pmatrix}$.
- Observe that $x^2 + x$ is the second basis vector in $\mathcal{A}$, so $[x^2 + x]_{\mathcal{A}} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(d)
- $[T]_{\mathcal{S}}^{\mathcal{S}} = \left([T(1)]_{\mathcal{S}}\ [T(x)]_{\mathcal{S}}\ [T(x^2)]_{\mathcal{S}}\right) = ([0]_{\mathcal{S}}\ [1]_{\mathcal{S}}\ [2x]_{\mathcal{S}}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.
- $[T]_{\mathcal{A}}^{\mathcal{S}} = [T]_{\mathcal{S}}^{\mathcal{S}}[I]_{\mathcal{A}}^{\mathcal{S}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
- $[T]_{\mathcal{S}}^{\mathcal{A}} = [I]_{\mathcal{S}}^{\mathcal{A}}[T]_{\mathcal{S}}^{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 1 \end{pmatrix}$.
- $[T]_{\mathcal{A}}^{\mathcal{A}} = [I]_{\mathcal{S}}^{\mathcal{A}}[T]_{\mathcal{A}}^{\mathcal{S}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -2 \\ 0 & 0 & 0 \\ 1/2 & 3/2 & 1 \end{pmatrix}$.

2. Consider the following vectors which interact nicely with this linear transformation:

$$\mathbf{b}_1 := \begin{pmatrix} -3 \\ 1 \end{pmatrix} \qquad \mathbf{b}_2 := \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Note that $\mathbf{b}_1$ is parallel to the line $y = -\frac{1}{3}x$, while $\mathbf{b}_2$ is perpendicular to it. Thus $T(\mathbf{b}_1) = \mathbf{b}_1$ while $T(\mathbf{b}_2 = -\mathbf{b}_2$. Consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Then by the observations we just made

$$[T]_{\mathcal{B}}^{\mathcal{B}} = ([T(\mathbf{b}_1)]_{\mathcal{B}}\ [T(\mathbf{b}_2]_{\mathcal{B}}) = ([\mathbf{b}_1]_{\mathcal{B}}\ [-\mathbf{b}_2]_{\mathcal{B}}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To compute $[T]_{\mathcal{S}}^{\mathcal{S}}$, we will need the change of coordinate matrices $[I]_{\mathcal{B}}^{\mathcal{S}}$ and $[I]_{\mathcal{S}}^{\mathcal{B}}$:

$$[I]_{\mathcal{B}}^{\mathcal{S}} = ([I(\mathbf{b}_1)]_{\mathcal{S}}\ [I(\mathbf{b}_2)]_{\mathcal{S}}) = (\mathbf{b}_1\ \mathbf{b}_2) = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}$$
$$[I]_{\mathcal{S}}^{\mathcal{B}} = \left([I]_{\mathcal{B}}^{\mathcal{S}}\right)^{-1} = \begin{pmatrix} -3/10 & 1/10 \\ 1/10 & 3/10 \end{pmatrix}$$

Finally, we compute

$$[T]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{B}}^{\mathcal{S}}[T]_{\mathcal{B}}^{\mathcal{B}}[I]_{\mathcal{S}}^{\mathcal{B}} = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3/10 & 1/10 \\ 1/10 & 3/10 \end{pmatrix} = \begin{pmatrix} 4/5 & -3/5 \\ -3/5 & -4/5 \end{pmatrix}$$

3. Suppose A and B are similar. Then there exists an invertible Q such that $A = Q^{-1}BQ$. Then using Exercise 6 from Homework 3 we have

$$\mathrm{Tr}(A) = \mathrm{Tr}(Q^{-1}BQ) = \mathrm{Tr}(BQQ^{-1}) = \mathrm{Tr}(B).$$

□

4. Let θ be the counter-clockwise angle between the positive x-axis and $\mathbf{v}_1$. Consider

$$R_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \frac{1}{\sqrt{a^2 + c^2}} \begin{pmatrix} a & c \\ -c & a \end{pmatrix}.$$

Recall that $R_{-\theta}$ is the linear transformation that rotates $\mathbb{R}^2$ by $-\theta$ radians. Observe that

$$R_{-\theta}\mathbf{v}_1 = \frac{1}{\sqrt{a^2+c^2}}\begin{pmatrix} a^2+c^2 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{a^2+c^2} \\ 0 \end{pmatrix}.$$

Also, we have

$$R_{-\theta}\mathbf{v}_2 = \frac{1}{\sqrt{a^2+c^2}}\begin{pmatrix} ab+cd \\ ad-bc \end{pmatrix}.$$

The parallelogram associated to $R_{-\theta}\mathbf{v}_1$ and $R_{-\theta}\mathbf{v}_2$ has area given by its base times its height (like the example from lecture):

$$\sqrt{a^2+c^2}\cdot\left|\frac{1}{\sqrt{a^2+c^2}}(ad-bc)\right| = |ad-bc|.$$

The parallelogram associated to $\mathbf{v}_1$ and $\mathbf{v}_2$ is a rotation of the above parallelogram and therefore has the same area. □

5. (a) We have

$$\det(A^TA) = \det(A^T)\det(A) = \det(A)\det(A^T) = \det(AA^T).$$

Thus $\det(A^TA) \neq 0$ if and only if $\det(AA^T) \neq 0$. Since the determinant being nonzero is equivalent to the matrix being invertible, we have A^TA is invertible if and only if AA^T is invertible. □

(b) Consider $A = (1\ \ 0) \in M_{1\times 2}$. Then

$$A^TA = \begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ \ 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Since this matrix is upper triangular, its determinant is the product of its diagonal entries: $\det(A^TA) = 0$. Thus A^TA is not invertible. On the other hand,

$$AA^T = (1\ \ 0)\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1\ \ 0) = (1),$$

which is the identity matrix in $M_{1\times 1}$ and in particular is invertible. □

6. (a) Observe that if E is elementary, then so is

$$\begin{pmatrix} E & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix}.$$

Moreover, both elementary matrices are of the same type. So if E is type 1, then both matrices have determinant -1. If E is type 2 with scalar $\alpha \neq 0$ on the diagonal, then so is the above matrix and hence they both have determinant α. Finally, if E is type 3 then so is the above matrix and hence both have determinant equal to one. □

(b) This follows by the same argument as in the previous part. □

(c) Let $E_1, \ldots, E_k$ be elementary matrices such that $A = E_1 \cdots E_k\tilde{A}$, where $\tilde{A}$ is the RREF of A. These exist since row operations can be used to go from $\tilde{A}$ to A. Note that: (i) $\tilde{A}$ is upper triangular; (ii) A is not invertible iff $\tilde{A}$ is missing pivot and hence has a zero along its diagonal; and (iii) A is invertible iff $\tilde{A} = I_n$.

Let $F_1, \ldots, F_\ell$ be elementary matrices such that $C = \tilde{C}F_\ell \cdots F_1$, where $\tilde{C}$ is upper triangular. To see that these exist, let D be the RREF of C^T. By using a series of row exchanges, one can turn D into a **lower** triangular matrix D' (simply reverse the order of rows). Thus there are elementary matrices $G_1, \ldots, G_\ell$ such that $C^T = G_1 \cdots G_\ell D'$. Taking the transpose of each side yields $C = (D')^T G_\ell^T \cdots G_1^T$. Since D' is lower triangular, $\tilde{C} := (D')^T$ is upper triangular, and $F_1 := G_1^T, \ldots, F_\ell := G_\ell^T$ are elementary matrices. Note that: (i) C is not invertible iff $\tilde{C}$ has a zero along its diagonal; and (ii) C is invertible iff $\tilde{C} = I_n$. Indeed, C is invertible iff C^T and then we infer (i) and (ii) from D.

Now, since all elementary matrices are invertible, we can define $\tilde{B} := (E_1 \cdots E_k)^{-1} B (F_\ell \cdots F_1)^{-1}$. Then $B = E_1 \cdots E_k \tilde{B} F_\ell \cdots F_1$. Observe that

$$\begin{aligned}\begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} &= \begin{pmatrix} E_1 \cdots E_k \tilde{A} & E_1 \cdots E_k \tilde{B} F_\ell \cdots F_1 \\ \mathbf{0} & \tilde{C} F_\ell \cdots F_1 \end{pmatrix} \\ &= \begin{pmatrix} E_1 & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix} \cdots \begin{pmatrix} E_k & \mathbf{0} \\ \mathbf{0} & I_n \end{pmatrix} \begin{pmatrix} \tilde{A} & \tilde{B} \\ \mathbf{0} & \tilde{C} \end{pmatrix} \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & F_\ell \end{pmatrix} \cdots \begin{pmatrix} I_m & \mathbf{0} \\ \mathbf{0} & F_1 \end{pmatrix}\end{aligned}$$

Since $\tilde{A}$ and $\tilde{C}$ are upper triangular, so is

$$M := \begin{pmatrix} \tilde{A} & \tilde{B} \\ \mathbf{0} & \tilde{C} \end{pmatrix}$$

Using the first two parts and the fact that the determinant of a product is the product of the determinants, we have

$$\det \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} = \det(E_1) \cdots \det(E_k) \det(M) \det(F_\ell) \cdots \det(F_1).$$

If either A or C is not invertible, then $\det(A)\det(C) = 0$ and as noted above either $\tilde{A}$ or $\tilde{C}$ will have a zero along their diagonal. But then so does M and hence $\det(M) = 0$. It follows that

$$\det \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} = 0 = \det(A)\det(C).$$

Finally, suppose A and C are both invertible. As noted above, $\tilde{A} = I_m$ and $\tilde{C} = I_n$. Therefore the diagonal entries of M are all equal to one so that $\det(M) = 1$. Also note that this implies $A = E_1 \cdots E_k$ and $C = F_\ell \cdots F_1$. Thus

$$\begin{aligned}\det \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix} &= \det(E_1) \cdots \det(E_k) \det(F_\ell) \cdots \det(F_1) \\ &= \det(E_1 \cdots E_k) \det(F_\ell \cdots F_1) = \det(A)\det(C),\end{aligned}$$

as claimed. $\square$