1. (a) **(10 pts)** We first find the RREF of A using row operations:

$$\begin{pmatrix} 2 & 0 & -2 & -1 & -7 \\ 0 & 1 & 2 & 1 & -2 \\ 1 & 1 & 1 & 0 & -5 \end{pmatrix} \overset{R3\leftrightarrow R1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & -5 \\ 0 & 1 & 2 & 1 & -2 \\ 2 & 0 & -2 & -1 & -7 \end{pmatrix}$$
$$\underset{R3\mapsto R3-2R1}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & -5 \\ 0 & 1 & 2 & 1 & -2 \\ 0 & -2 & -4 & -1 & 3 \end{pmatrix}$$
$$\underset{R3\mapsto R3+2R2}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & -5 \\ 0 & 1 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$
$$\underset{R2\mapsto R2-R3}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & -5 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$
$$\underset{R1\mapsto R1-R2}{\longrightarrow} \begin{pmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Since the pivots are in columns 1, 2, and 4 it follows that the corresponding columns of A:

$$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

form a basis for the column space of A.

(b) **(3 pts)** Since the pivots of the RREF of A appear in rows 1, 2, and 3 those rows form a basis for the row space of A:

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ -4 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \\ -1 \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}.$$

(c) **(4 pts)** Using the RREF of A we see that $A\mathbf{x} = \mathbf{0}$ has solutions of the form

$$\mathbf{x} = \begin{pmatrix} x_3 + 4x_5 \\ -2x_3 + x_5 \\ x_3 \\ x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \qquad x_3, x_5 \in \mathbb{R}.$$

Thus

$$\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 4 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

is a basis for $\mathrm{Ker}(A)$.

(d) **(3 pts)** By the rank-nullity theorem, we know

$$\dim(\mathrm{Ker}(A^T)) = \mathrm{nullity}(A^T) = 3 - \mathrm{rank}(A^T).$$

By part (b), we have that $\mathrm{rank}(A^T) = 3$ and so $\dim(\mathrm{Ker}(A^T)) = 0$.

2. (a) **(3 pts)** Recall that the jth column of $[I]_{\mathcal{B}}^{\mathcal{S}}$ is given by

$$[I(\mathbf{v}_j)]_{\mathcal{S}} = [\mathbf{v}_j]_{\mathcal{S}} = \mathbf{v}_j.$$

Thus

$$[I]_{\mathcal{B}}^{\mathcal{S}} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

(b) **(10 pts)** We know $[I]_{\mathcal{S}}^{\mathcal{B}} = ([I]_{\mathcal{B}}^{\mathcal{S}})^{-1}$, so we compute the inverse by performing row operations on $([I]_{\mathcal{B}}^{\mathcal{S}} \mid I_3)$:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \underset{R1\mapsto R1-2R3}{\overset{R1\leftrightarrow R_3}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & -1 & 1 & 0 & -2 \end{array}\right)$$

$$\underset{R3\mapsto R3+2R2}{\overset{R1\mapsto R1-R2}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

$$\underset{R2\mapsto R2-R3}{\overset{R1\mapsto R1+R3}{\longrightarrow}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 2 & -2 \end{array}\right)$$

Thus

$$[I]_{\mathcal{S}}^{\mathcal{B}} = \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

(c) **(7 pts)** We are told $\mathcal{B}$ is a basis of eigenvectors of A. Thus

$$[A]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

So using a change of basis we see that

$$A = [A]_{\mathcal{S}}^{\mathcal{S}} = [I]_{\mathcal{B}}^{\mathcal{S}}[A]_{\mathcal{B}}^{\mathcal{B}}[I]_{\mathcal{S}}^{\mathcal{B}} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ -1 & -1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ -3 & -3 & 6 \\ 3 & 6 & -6 \end{pmatrix} = \begin{pmatrix} -3 & -6 & 6 \\ 0 & 3 & 0 \\ -3 & -3 & 6 \end{pmatrix}$$

3. (a) **(7 pts)** We will compute the determinant using cofactor expansion along the second row:

$$\mathrm{char}_A(z) = \det(A - zI) = \det\begin{pmatrix} -3-z & -6 & 6 \\ 0 & 3-z & 0 \\ -3 & -3 & 6-z \end{pmatrix} = 0 + (-1)^{2+2}(3-z)((-3-z)(6-z) - 18) + 0$$
$$= (3-z)(-18 - 3z + z^2 - 18) = (3-z)(z^2 - 3z) = -z(z-3)^2.$$

(b) **(3 pts)** Clearly the roots of $\mathrm{char}_A(z)$ are $z = 0$ and $z = 3$. Thus $\sigma(A) = \{0, 3\}$.

(c) **(10 pts)** From the characteristic polynomial we know the algebraic multiplicities: $m_0(A) = 1$ and $m_3(A) = 2$. Thus $\mathrm{Ker}(A - 0I)$ is at most one-dimensional and from the previous question we see that $(2,0,1)^T$ is an eigenvector with eigenvalue 0. Thus $(2,0,1)^T$ forms a basis for $\mathrm{Ker}(A - 0I)$. Also we know $\mathrm{Ker}(A - 3I)$ is at most two-dimensional and from the previous problem we see that $(0,1,1)^T$ and $(-1,1,0)^T$ are linearly independent vectors in this eigenspace. Thus they necessarily form a basis for the eigenspace.

Alternatively, we can compute direclty. For $\lambda = 0$, we compute the kernel of $A - 0I = A$:

$$\left(\begin{array}{ccc|c} -3 & -6 & 6 & 0 \\ 0 & 3 & 0 & 0 \\ -3 & -3 & 6 & 0 \end{array}\right) \xrightarrow[\substack{R2\mapsto\frac{1}{3}R2 \\ R3\mapsto-\frac{1}{3}R3}]{R1\mapsto-\frac{1}{3}R1} \left(\begin{array}{ccc|c} 1 & 2 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right)$$

$$\xrightarrow[R3\mapsto R3-R1]{R1\mapsto R1-2R2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R3\mapsto R3+R2]{} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $(2,0,1)^T$ is a basis for $\mathrm{Ker}(A - 0I)$.
For $\lambda = 3$, we compute the kernel of $A - 3I$:

$$\left(\begin{array}{ccc|c} -6 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 0 \end{array}\right) \xrightarrow[R3\mapsto 2R3-R1]{R1\mapsto-\frac{1}{6}R1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Thus $(-1,1,0)^T$ and $(1,0,1)^T$ form a basis for $\mathrm{Ker}(A - 3I)$.

4. (a) **(3 pts)** For each $i = 2, \ldots, n$ we have

$$A(\mathbf{w}_i) = A(\mathbf{v}_1 - \mathbf{v}_i) = A\mathbf{v}_1 - A\mathbf{v}_i = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Thus $\mathbf{w}_i \in \mathrm{Ker}(A)$ for $i = 2, \ldots, n$. □

(b) **(7 pts)** Suppose

$$\alpha_2\mathbf{w}_2 + \cdots + \alpha_n\mathbf{w}_n = \mathbf{0}$$

for scalars $\alpha_2, \ldots, \alpha_n$. Using $\mathbf{w}_i = \mathbf{v}_1 - \mathbf{w}_i$, we obtain

$$\mathbf{0} = \alpha_2(\mathbf{v}_1 - \mathbf{v}_2) + \cdots + \alpha_n(\mathbf{v}_1 - \mathbf{v}_n) = (\alpha_2 + \cdots + \alpha_n)\mathbf{v}_1 - \alpha_2\mathbf{v}_2 - \cdots - \alpha_n\mathbf{v}_n.$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are a basis and consequently linearly independent, we must have that the above coefficients are all zero. In particular, $-\alpha_2 = \cdots = -\alpha_n = 0$ which of course implies $\alpha_2 = \cdots = \alpha_n = 0$. Thus $\mathbf{w}_2, \ldots, \mathbf{w}_n$ are linearly independent. □

(c) **(10 pts)** We claim that $\mathrm{rank}(A) = 1$. Indeed, for any $\mathbf{v} \in \mathbb{R}^n$

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n$$

for some scalars $\alpha_1, \ldots, \alpha_n$. Using the linearity of A we then have

$$A\mathbf{v} = \alpha_1 A\mathbf{v}_1 + \cdots + \alpha_n A\mathbf{v}_n = \alpha_1\mathbf{b} + \cdots + \alpha_n\mathbf{b} = (\alpha_1 + \cdots + \alpha_n)\mathbf{b}.$$

Thus $A\mathbf{v} \in \mathrm{span}\{\mathbf{b}\}$. Thus $\mathrm{Ran}(A) \subset \mathrm{span}\{\mathbf{b}\}$ and the reverse inclusion follows since $A(\alpha\mathbf{v}_1) = \alpha\mathbf{b}$. Thus $\mathrm{Ran}(A) = \mathrm{span}\{\mathbf{b}\}$ and in particular $\mathrm{rank}(A) = \dim(\mathrm{Ran}(A)) = 1$. Now, by the rank-nullity theorem, we have

$$\dim(\mathrm{Ker}(A)) = \mathrm{nullity}(A) = n - \mathrm{rank}(A) = n - 1.$$

Thus $\mathbf{w}_2, \ldots, \mathbf{w}_n$ is a set of $n-1$ linearly independent vectors in a subspace with dimension equal to $n-1$. It follows that $\mathbf{w}_2, \ldots, \mathbf{w}_n$ is necessarily a basis for $\mathrm{Ker}(A)$. □

5. (a) **(2 pts)** Every invertible matrix can be written as a ~~sum~~ product of elementary matrices.

(b) **(2 pts)** Any two matrix representations of a linear transformation are ~~equal~~ similar.

(c) **(2 pts)** A generating system for a finite-dimensional vector space V cannot have ~~more~~ fewer than $\dim(V)$ vectors in it.

(d) **(2 pts)** A vector space is finite-dimensional if and only if it has a ~~unique~~ basis consisting of finitely many vectors.

(e) **(2 pts)** The ~~nullity~~ rank of a matrix and its transpose are equal.

(f) **(2 pts)**The ~~row~~ column space of a matrix A is equal to the range of A.

(g) **(2 pts)** The determinant is invariant under row ~~reordering~~ replacement.

(h) **(2 pts)** The determinant is ~~linear~~ multiplicative.

(i) **(2 pts)** For a linear transformation $T\colon V \to V$, an eigenvector of T is a non-zero vector $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \lambda\mathbf{v}$ for some scalar λ.

(j) **(2 pts)** For a linear transformation $T\colon V \to V$ with eigenvalue λ, the dimension of $\text{Ker}(T - \lambda I)$ is the geometric multiplicity of λ.