3.1 Signed Measures

Def Let $(X,\mathcal{M})$ be a measurable space. A signed measure on $(X,\mathcal{M})$ is a function $\nu:\mathcal{M}\to[-\infty,\infty]$ such that

1. $\nu(\emptyset)=0$
2. ν assumes at most one of the values $\pm\infty$
3. If $(E_n)_{n\in\mathbb{N}}\subset\mathcal{M}$ is a disjoint collection, then
$$\nu\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\nu(E_n)$$
where the series converges absolutely if the quantity on the left is finite. $\square$

Every measure is a signed measure, and for emphasis we may call a measure a positive measure.

Ex ① Let μ_1,μ_2 be measures on common measurable space, at least one which is finite. Then $\nu(E):=\mu_1(E)-\mu_2(E)$ is a signed measure. For $(E_n)_{n\in\mathbb{N}}\subset\mathcal{M}$ disjoint, if $\nu(\bigcup E_n)$ is finite, then necessarily $\mu_j(\bigcup E_n)<\infty$ for $j=1,2$. Consequently
$$\sum_{n=1}^{\infty}|\nu(E_n)|\le\sum_{n=1}^{\infty}\mu_1(E_n)+\mu_2(E_n)=\mu_1\left(\bigcup_{n=1}^{\infty}E_n\right)+\mu_2\left(\bigcup_{n=1}^{\infty}E_n\right)<\infty$$
so the series $\sum_{n=1}^{\infty}\nu(E_n)$ converges absolutely.

② Let $(X,\mathcal{M},\mu)$ be a measure space and let $f:X\to\overline{\mathbb{R}}$ be an $\mathcal{M}$-measurable function such that at least one of $\int f^+d\mu$ or $\int f^-d\mu<\infty$ is finite. Then $\nu(E):=\int_E f\,d\mu$ is a signed measure. In fact, $\nu=\mu_+-\mu_-$ where $\mu_+(E)=\int_E f^+d\mu$ and $\mu_-(E)=\int_E f^-d\mu$ are measures by Exercise 3 on Homework 5. $\square$

We will see that these are really the only examples of signed measures.

The exact same proofs that gave continuity from above and below for positive measures (Theorem 1.8) also give the following.

Proposition 3.1 Let ν be a signed measure on $(X,\mathcal{M})$. If $E_1\subset E_2\subset\cdots$ with $E_n\in\mathcal{M}$, $n\in\mathbb{N}$ then
$$\nu\left(\bigcup_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\nu(E_n)$$
If $E_1\supset E_2\supset\cdots$ with $E_n\in\mathcal{M}$, $n\in\mathbb{N}$, and if $\nu(E_1)$ is finite then
$$\nu\left(\bigcap_{n=1}^{\infty}E_n\right)=\lim_{n\to\infty}\nu(E_n)$$

Def Let ν be a signed measure on $(X,\mathcal{M})$. We call $E\in\mathcal{M}$ positive (resp. negative) for ν if $\nu(F)\ge0$ (resp. $\nu(F)\le0$) for all $F\subset E$ with $F\in\mathcal{M}$. We call E a ν-null set if $\nu(F)=0$ for all $F \subset E$ with $F \in \mathcal{M}$.

Remark If ν happens to be a positive measure, then a set $E \in \mathcal{M}$ is ν-null in the above sense iff $\nu(E)=0$, since monotonicity holds for positive measures.

Ex Let $\nu(E) = \int_E f\,d\mu$ be as in Example 2 above. Then $E \in \mathcal{M}$ is
- positive for ν iff $f \geq 0$ μ-a.e. on E
- negative for ν iff $f \leq 0$ μ-a.e. on E
- ν-null iff $f=0$ μ-a.e. on E.

Lemma 3.2 Let ν be a signed measure on $(X,\mathcal{M})$.

① If $E \in \mathcal{M}$ is positive for ν, then every $F \subset E$ with $F \in \mathcal{M}$ is positive for ν.

② If $E_n \in \mathcal{M}$ is positive for ν for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n$ is positive for ν.

Proof ①: This follows from the definition of being positive for ν.

②: For each $n \in \mathbb{N}$, let $F_n := E_n \setminus (E_1 \cup \cdots \cup E_n)$. Then $F_n \subset E_n$ is positive for ν by ①, and $(F_n)_{n\in\mathbb{N}}$ is a disjoint collection. Now if

$$F \subset \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n,$$

then

$$\nu(F) = \nu\Big(\bigcup_{n=1}^{\infty} F \cap F_n\Big) = \sum_{n=1}^{\infty} \nu(F \cap F_n) \geq 0$$

Hence $\bigcup_{n=1}^{\infty} E_n$ is positive for ν.

Theorem 3.3 (The Hahn Decomposition Theorem) If ν is a signed measure on $(X,\mathcal{M})$, then there exists a partition $X = P \cup N$ so that P is positive for ν and N is negative for ν. If $X = P' \cup N'$ is another such partition, then $P \Delta P'$ ($= N \Delta N'$) is ν-null.

Proof Without loss of generality, we may assume ν does not take the value $+\infty$ (otherwise work with $-\nu$). Let

$$R := \sup\{\nu(E) : E \in \mathcal{M} \text{ is positive for } \nu\}.$$

Then we can find a sequence $(E_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ of positive sets satisfying $\nu(E_n) \to R$. Set $P := \bigcup_{n=1}^{\infty} E_n$, which is positive for ν by Lemma 3.2.②, and $\nu(P) = R$ by Lemma 3.1. Thus $R < \infty$.
We claim $N := X \setminus P$ is negative for ν. First note N cannot contain any non-ν-null positive sets: if $E \subset N$ is positive then $E \cup P$ is positive with $\nu(E \cup P) = \nu(E) + \nu(P) > R$, a contradiction. Consequently, if $A \subset N$ satisfies $\nu(A) > 0$, then there must exist $B \subset A$ with $\nu(B) < 0$. Note that

$$\nu(A \setminus B) = \nu(A) - \nu(B) > \nu(A)$$

Hence any $A \subset N$ with $\nu(A) > 0$ has a subset with strictly larger measure. Now, suppose towards a contradiction that N is not negative for ν. Then N has subsets of strictly positive measure. Let $n_1 \in \mathbb{N}$ be the smallest integer such that $\exists B \subset N$ with $\nu(B) > \frac{1}{n_1}$, and let $A_1 := B$. We inductively define $(n_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ and $(A_k)_{k\in\mathbb{N}} \subset N$ as follows. Let $n_{k+1} \in \mathbb{N}$ be the smallest integer such that $\exists B \subset A_k$ with $\nu(B) > \nu(A_k) + \frac{1}{n_{k+1}}$ (which

exists by our above observation) and let $A_{k+1} := B$. Note that $\nu(A_{k+1}) > \nu(A_k) + \frac{1}{n_{k+1}} > \cdots > \sum_{j=1}^{k+1} \frac{1}{n_k}$.
Denote $A := \bigcap_{k=1}^{\infty} A_k$, then by Proposition 3.1 we have

$$\infty > \nu(A) = \lim_{k\to\infty} \nu(A_k) > \sum_{k=1}^{\infty} \frac{1}{n_k}$$

Thus we must have $n_k \to \infty$. Since A has a subset with strictly larger measure, $\exists\, B \subset A$ and $n \in \mathbb{N}$ with $\nu(B) > \nu(A) + \frac{1}{n}$. Let $k \in \mathbb{N}$ be large enough so that $n < n_k$ and

$$\nu(B) > \nu(A_{k-1}) + \tfrac{1}{n}.$$

But then $B \subset A \subset A_{k-1}$ and $n < n_k$ contradict the definition of n_k. Thus N must be negative.
Finally, let $X = P' \cup N'$ be another partition such that P' is positive for ν and N' is negative for ν. Then $P \setminus P' \subset P$ and $P \setminus P' = P \cap (P')^c = P \cap N' \subset N'$. Thus $P \setminus P'$ is both positive and negative for ν, and is therefore ν-null. Similarly for $P' \setminus P$. Hence $P \Delta P' = P \setminus P' \cup P' \setminus P$ is ν-null. □

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Def Let ν be a signed measure on $(X, \mathcal{M})$. A partition $X = P \cup N$ such that P is positive for ν and N is negative for ν is called a Hahn decomposition for ν □

While a Hahn decomposition is only unique up to ν-null sets by Theorem 3.3, it does lead to a unique representation of ν as a difference of two positive measures (see Theorem 3.4).

Def Given two signed measures μ and ν on $(X, \mathcal{M})$, we say μ and ν are mutually singular if there exists a partition $X = E \cup F$ such that E is μ-null and F is ν-null. In this case, we write: $\mu \perp \nu$. □

Theorem 3.4 (The Jordan Decomposition Theorem) For a signed measure ν on $(X, \mathcal{M})$, there exist unique positive measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$.

Proof Let $X = P \cup N$ be a Hahn decomposition for ν and define

$$\nu^+(E) := \nu(E \cap P)$$
$$\nu^-(E) := -\nu(E \cap N).$$

Then $\nu(E) = \nu^+(E) - \nu^-(E)$ since $E = (E \cap P) \cup (E \cap N)$ and the union is disjoint. Also, $\nu^+(N) = 0$ so N is ν^+-null since ν^+ is a positive measure. Similarly P is ν^--null, and so $\nu^+ \perp \nu^-$.
Now suppose μ^+ and μ^- are another pair of positive measures satisfying $\nu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. The latter condition implies there is a partition $X = E \cup F$ such that E is μ^+-null and F is μ^--null. But then for all $A \subset E$ we have

$$\nu(A) = \nu^+(A) - \nu^-(A) = \nu^+(A) \geq 0.$$

So E is positive for ν. Similarly, F is negative for ν. Hence $X = E \cup F$ is a Hahn decomposition for ν. So $P \Delta E = N \Delta F$ is ν-null by the Hahn-decomposition theorem (Theorem 3.3). Consequently for any $A \in \mathcal{M}$ we have

$$\nu^+(A) = \nu(A \cap P) = \nu(A \cap E) = \mu^+(A \cap E) - \mu^-(\cancel{A \cap E}) = \mu^+(A).$$

Similarly $\nu^- = \mu^-$. □

Def Let ν be a signed measure on $(X,\mathcal{M})$. The unique positive measures ν^+ and ν^- on $(X,\mathcal{M})$ satisfying $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ are called the <u>positive</u> and <u>negative variations</u> of ν, respectively, and $\nu = \nu^+ - \nu^-$ is called the <u>Jordan decomposition</u> of ν. The positive measure

$$|\nu| := \nu^+ + \nu^-$$

on $(X,\mathcal{M})$ is called the <u>total variation</u> of ν. $\square$

EX Let $(X,\mathcal{M},\mu)$ be a measure space and let $f: X \to \overline{\mathbb{R}}$ be $\mathcal{M}$-measurable such that at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. Then for the signed measure

$$\nu(E) = \int_E f \, d\mu,$$

$X = \{x \in X : f(x) \geq 0\} \cup \{x \in X : f(x) < 0\}$ is a Hahn decomposition for ν. One can also move any measurable subset of $f^{-1}(\{0\})$ over. The positive and negative variations of ν are

$$\nu^{\pm}(E) = \int_E f^{\pm}(E) \, d\mu,$$

and

$$|\nu|(E) = \int_E |f| \, d\mu$$

is the total variation of ν.

Conversely, if ν is any signed measure on $(X,\mathcal{M})$ with a Hahn decomposition $X = P \cup N$, then for $E \in \mathcal{M}$

$$\int_E 1_P - 1_N \, d|\nu| = |\nu|(E \cap P) - |\nu|(E \cap N) = \nu^+(E) - \nu^-(E) = \nu(E)$$

Thus $\nu = \int_E f \, d\mu$ where $f = 1_P - 1_N$ and $\mu = |\nu|$. $\square$

Note that ν^+ is finite iff ν omits the value ∞, ν^- is finite iff ν omits the value $-\infty$, and $|\nu|$ is finite iff ν omits both $\pm\infty$.

Def Let ν be a signed measure on $(X,\mathcal{M})$. We say ν is <u>finite</u> (resp. <u>σ-finite</u>) if $|\nu|$ is finite (resp. σ-finite). We define

$$L^1(X,\nu) := L^1(X,\nu^+) \cap L^1(X,\nu^-)$$

and for $f \in L^1(X,\nu)$ we define

$$\int_X f \, d\nu := \int_X f \, d\nu^+ - \int_X f \, d\nu^-$$

$\square$

EX Let $(X,\mathcal{M},\mu)$ be a measure space. For an $\mathcal{M}$-measurable f, $\nu(E) = \int_E f \, d\mu$ is finite iff

$$|\nu|(E) = \int_E |f| \, d\mu$$

is finite iff $f \in L^1(X,\mu)$. For $g \in L^1(X,\nu)$, Exercise 3 on Homework 5 implies

$$\int_X g \, d\nu = \int_X g \, d\nu^+ - \int g \, d\nu^- = \int_X g f^+ d\mu - \int_X g f^- d\mu = \int_X g f \, d\mu.$$

$\square$

3.2 The Lebesgue-Radon-Nikodym Theorem

Def Given two signed measures μ and ν on $(X,\mathcal{M})$, we say ν is absolutely continuous with respect to μ if whenever $E\in\mathcal{M}$ is μ-null it is also ν-null. In this case we write: $\nu\ll\mu$. □

Using the (positive, negative, total) variations of ν and μ one can define absolute continuity in many equivalent ways:

$$\begin{array}{ccccc}
 & \nu\ll\mu & \overset{2}{\Longleftrightarrow} & \nu\ll|\mu| & \underset{3}{\overset{\Longleftarrow}{\not\Longrightarrow}} \ \nu\ll\mu^+ \text{ and } \nu\ll\mu^- \\
 & 1\ \Updownarrow & & \Updownarrow & \\
\nu^+\ll\mu \text{ and } \nu^-\ll\mu \ \Longleftrightarrow & |\nu|\ll\mu & \Longleftrightarrow & |\nu|\ll|\mu| &
\end{array}$$

We leave most of these as exercises, but prove a few here:

1: Suppose $\nu\ll\mu$. Let $X=P\cup N$ be a Hahn decomposition for ν. If $E\in\mathcal{M}$ is μ-null, then so are $E\cap P$ and $E\cap N$. Hence

$$|\nu|(E)=\nu^+(E)+\nu^-(E)=\nu(E\cap P)-\nu(E\cap N)=0-0=0$$

So E is $|\nu|$-null and $|\nu|\ll\mu$.
Conversely, if $|\nu|\ll\mu$ and $E\in\mathcal{M}$ is μ-null, then $|\nu|(F)=0$ for any $F\subset E$ with $F\in\mathcal{M}$. Consequently, $\nu^\pm(F)=0$ and so $\nu(F)=0$. Thus E is ν-null and $\nu\ll\mu$. □

2: If $\nu\ll\mu$ and $E\in\mathcal{M}$ is $|\mu|$-null, then the above argument shows that E is then μ-null. Hence E is ν-null and $\nu\ll|\mu|$.
Conversely, suppose $\nu\ll|\mu|$. Let $X=P\cup N$ be a Hahn decomposition for μ. If E is μ-null, then so are $E\cap P$ and $E\cap N$. Consequently

$$|\mu|(E)=\mu^+(E)+\mu^-(E)=\mu(E\cap P)-\mu(E\cap N)=0$$

So E is $|\mu|$-null and therefore ν-null. Thus $\nu\ll\mu$ □

3: One has $|\mu|\ll|\mu|$ by definition, but $|\mu|\ll\mu^+$ fails if μ^- is non-trivial. non-trivial. Indeed, let $X=P\cup N$ be a Hahn decomposition for μ. Since μ^- is non-trivial, $\mu^-(N)>0$ and hence $|\mu|(N)\geq\mu^-(N)>0$. However, $\mu^+(N)=0$. So $|\mu|\not\ll\mu^+$. Similarly, $|\mu|\not\ll\mu^-$ if μ^+ is non-trivial. □

In light of the above equivalences (namely $\nu\ll\mu\Leftrightarrow\nu\ll|\mu|$), we will usually assume μ is positive.

Another important observation is that $\nu\ll\mu$ and $\nu\perp\mu$ imply $\nu\equiv0$. Indeed, let $X=E\cup F$ be a decomposition such that E is ν-null and F is μ-null. Then for any $A\in\mathcal{M}$,

$$\nu(A)=\nu(A\cap E)+\nu(A\cap F).$$

The former term is zero since E is ν-null, and the latter is zero since $\mu(A \cap F) = 0$

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Theorem 3.5 Let ν be a finite signed measure and μ a positive measure on $(X, \mathcal{M})$. Then $\nu \ll \mu$ if and only if $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $|\nu(E)| < \varepsilon$ whenever $\mu(E) < \delta$.

Proof As observed above, $\nu \ll \mu$ iff $|\nu| \ll \mu$. Since $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$, it suffices to prove the theorem for $|\nu|$, and so we may just assume ν is a positive measure.

($\Longrightarrow$) We proceed by contrapositive and assume the ε-δ condition fails; that is, there exists $\varepsilon > 0$ such that for any $\delta > 0$ there exists $E \in \mathcal{M}$ satisfying $\mu(E) < \delta$ but $\nu(E) \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$ we can find $E_n \in \mathcal{M}$ with $\mu(E_n) < 2^{-n}$ and $\nu(E_n) \geq \varepsilon$. Define

$$F_N := \bigcup_{n=N}^{\infty} E_n$$

so that $\mu(F_N) < 2^{-N+1}$ and $F_N \supset F_{N+1}$. Letting $F := \bigcap_{N=1}^{\infty} F_N$, we therefore have $\mu(F) = 0$ by continuity from above. On the other hand

$$\nu(F_N) \geq \nu(E_N) \geq \varepsilon$$

for each $N \in \mathbb{N}$. Since ν is finite, continuity from above implies $\nu(F) \geq \varepsilon$. Thus ν is not absolutely continuous with respect to μ.

($\Longleftarrow$): Assume the ε-δ condition holds. For each $\varepsilon = \frac{1}{n}$, let $\delta_n > 0$ be the corresponding quantity. If $\mu(E) = 0$, then $\mu(E) < \delta_n$ and so $\nu(E) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence $\nu(E) = 0$. □

Def Let $(X, \mathcal{M}, \mu)$ be a measure space. We say an $\mathcal{M}$-measurable function $f : X \to \overline{\mathbb{R}}$ is an extended μ-integrable function if at least one of $\int f^+ d\mu$ or $\int f^- d\mu$ is finite. We write $d\nu = f d\mu$ to mean ν is the signed measure defined by $\nu(E) = \int_E f d\mu$. □

If $d\nu = f d\mu$, then $\nu \ll \mu$ since if $\mu(E) = 0$ then $1_E f = 0$ μ-a.e. and so

$$\nu(E) = \int_E f d\mu = \int_X 1_E f d\mu = 0$$

by Proposition 2.16. Also recall that ν is finite iff $f \in L^1(X, \mu)$. Thus the previous Theorem yields the following Corollary:

Corollary 3.6 Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f \in L^1(X, \mu)$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \int_E f d\mu \right| < \varepsilon$$

whenever $\mu(E) < \delta$.

Remark Exercise 1 on Homework 6 asked you to show $t \mapsto \int_{[a,t]} f dm$ was continuous for $f \in L^1(\mathbb{R}, m)$. Corollary 3.6 offers a very easy proof of this. □

$\boxed{\text{Lemma 3.7}}$ Let ν and μ be finite measures on $(X,\mathcal{M})$. Then either $\nu \perp \mu$ or there exists $\varepsilon > 0$ and $E \in \mathcal{M}$ such that $\mu(E) > 0$ and E is a positive set for $\nu - \varepsilon\mu$.

Proof For each $n \in \mathbb{N}$, let $X = P_n \cup N_n$ be a Hahn decomposition for $\nu - \frac{1}{n}\mu$. Define

$$P := \bigcup_{n=1}^{\infty} P_n \qquad \text{and} \qquad N := \bigcap_{n=1}^{\infty} N_n = P^c$$

Then N is a negative set for each $\nu - \frac{1}{n}\mu$. In particular, $\nu(N) \leq \frac{1}{n}\mu(N)$ all $n \in \mathbb{N}$, and hence $\nu(N) = 0$. If $\mu(P) = 0$, then $\nu \perp \mu$. Otherwise $\mu(P) > 0$ and thus $\mu(P_n) > 0$ for some sufficiently large $n \in \mathbb{N}$. Take $E := P_n$. $\square$

$\boxed{\text{Lemma 3.8}}$ Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(\nu_n)_{n\in\mathbb{N}}$ be signed measures on $(X,\mathcal{M})$ such that $\nu := \sum_{n=1}^{\infty} \nu_n$ defines a signed measure.

1. If $\nu_n \perp \mu$ for all $n \in \mathbb{N}$, then $\nu \perp \mu$
2. If $\nu_n \ll \mu$ for all $n \in \mathbb{N}$, then $\nu \ll \mu$.

Proof 1: For each $n \in \mathbb{N}$, let $X = E_n \cup F_n$ be a partition such that E_n is ν_n-null and F_n is μ-null. Then $E := \bigcap_{n=1}^{\infty} E_n$ is ν_n-null for all $n \in \mathbb{N}$ and hence ν-null. Also $F := \bigcup_{n=1}^{\infty} F_n$ is μ-null by countable subadditivity. Thus $\nu \perp \mu$.

2: If $E \in \mathcal{M}$ is μ-null, then for any measurable $F \subset E$, $\nu(F) = \sum_{n=1}^{\infty} \nu_n(F) = 0$. So $\nu \ll \mu$. $\square$

$\boxed{\text{Theorem 3.9}}$ (The Lebesgue–Radon–Nikodym Theorem) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For σ-finite signed measure ν on $(X, \mathcal{M})$, there exists unique σ-finite signed measures λ, ρ on $(X,\mathcal{M})$ satisfying

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ and $\rho \ll \mu$. Moreover, there exists an extended μ-integrable function $f: X \to \mathbb{R}$ such that $d\rho = f\,d\mu$, and f is unique up to μ-almost everywhere equivalence.

Proof First suppose ν and μ are finite positive measures. Consider:

$$\mathcal{F} := \{ f : X \to [0,\infty] : \int_E f\,d\mu \leq \nu(E) \quad \forall E \in \mathcal{M}\}$$

Then $\mathcal{F} \neq \emptyset$ since $0 \in \mathcal{F}$. Also if $f, g \in \mathcal{F}$, then $\max(f,g) \in \mathcal{F}$: if $A := \{x \in X : f(x) > g(x)\}$ then for $E \in \mathcal{M}$

$$\int_E \max(f,g)\,d\mu = \int_{E\cap A} f\,d\mu + \int_{E\setminus A} g\,d\mu \leq \nu(E\cap A) + \nu(E\setminus A) = \nu(E).$$

Let $a := \sup\{\int f\,d\mu : f \in \mathcal{F}\}$, and let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ be such that $\int f_n\,d\mu \to a$. By replacing f_n with $\max(f_1, \ldots, f_n)$ we may — by our above observation — assume $f_n \leq f_{n+1}$. Then the monotone convergence theorem implies $f := \sup f_n \in \mathcal{F}$ with $\int f\,d\mu = a$. Since $a \leq \nu(X) < \infty$, we know f is finite μ-a.e. by Proposition 2.20, and so considering $f 1_{\{x : f(x) < \infty\}}$ if necessary, we may assume f is finitely valued.

Now, the measure $d\lambda := d\nu - f\,d\mu$ is positive since $f \in \mathcal{F}$. If $\lambda \not\perp \mu$, then using Lemma 3.7 we can find $\varepsilon > 0$ and $E \in \mathcal{M}$ so that $\mu(E) > 0$ and E is positive for $\lambda - \varepsilon\mu$.

Consider $g := f + \varepsilon 1_E$. We have for $F \in \mathcal{M}$

$$\int_F g\,d\mu = \int_F f\,d\mu + \varepsilon \int_F 1_E\,d\mu$$
$$= \int_F f\,d\mu + \varepsilon\,\mu(E\cap F)$$
$$\leq \int_F f\,d\mu + \lambda(E\cap F)$$
$$\leq \int_F f\,d\mu + \lambda(F) = \nu(F)$$

Hence $g \in \mathcal{F}$, but

$$\int_X g\,d\mu = \int_X f\,d\mu + \varepsilon \int_X 1_E\,d\mu = a + \varepsilon\,\mu(E) > a,$$

contradicting the definition of a. Thus we must have $\lambda \perp \mu$. Letting $d\rho := f\,d\mu$, we have shown existence of the claimed measures. Towards showing uniqueness, suppose $\nu = \lambda' + \rho'$ for $\lambda' \perp \mu$ and $\rho' \ll \mu$ with $d\rho' = f'\,d\mu$. Then $\lambda + \rho = \nu = \lambda' + \rho'$ implies $\lambda - \lambda' = \rho' - \rho$, and by Lemma 3.8 we have $(\lambda - \lambda') \perp \mu$ and $(\rho' - \rho) \ll \mu$. Hence $\lambda - \lambda' = \rho - \rho' = 0$ by our observation preceding Theorem 3.5. In particular, $\int_E f\,d\mu = \int_E f'\,d\mu$ for all $E \in \mathcal{M}$, and so $f = f'$ μ-a.e. by Proposition 2.23.

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Next, assume ν and μ are σ-finite positive measures. By taking intersections of the respective finite-measure decompositions of X for ν and μ, we have $X = \bigsqcup_{n=1}^{\infty} X_n$ where $\nu(X_n), \mu(X_n) < \infty$ for each $n \in \mathbb{N}$. Define $\nu_n(E) := \nu(E \cap X_n)$ and $\mu_n(E) := \mu(E\cap X_n)$. The first part of the proof yields $\lambda_n \perp \mu_n$ and $d\rho_n = f_n\,d\mu_n$ such that $\nu_n = \lambda_n + \rho_n$. Since $\nu_n(X_n^c) = \mu_n(X_n^c) = 0$, we have

$$\lambda_n(X_n^c) = \nu_n(X_n^c) - \rho_n(X_n^c) = 0,$$

and replacing f_n with $f_n 1_{X_n}$ if necessary, we may assume $f_n = 0$ on X_n^c. Define

$$\lambda := \sum_{n=1}^{\infty} \lambda_n, \quad \rho := \sum_{n=1}^{\infty} \rho_n, \quad \text{and} \quad f := \sum_{n=1}^{\infty} f_n.$$

Then $\lambda \perp \mu$ and $\rho \ll \mu$ by Lemma 3.8. These measures are also σ-finite since $\lambda(X_n) = \lambda_n(X_n)$ and $\rho(X_n) = \rho_n(X_n)$ are finite. Also, $f \geq 0$ and so is automatically an extended μ-integrable function, and the MCT (Theorem 2.14) implies $d\rho = f\,d\mu$. Uniqueness follows as in the first part of the proof.

Finally, for ν a σ-finite signed measure, apply the previous part to $\nu^{\pm}$ and take the differences of the resulting measures. □

Remark Note that in the above proof, if ν was positive then $f \geq 0$. □

Def Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space and let ν be a σ-finite signed measure on $(X, \mathcal{M})$. The <u>Lebesgue decomposition</u> of ν with respect to μ is

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ and $\rho \ll \mu$. If $\nu \ll \mu$ so that $d\nu = f\,d\mu$ for some extended μ-integrable function, then we call f the <u>Radon–Nikodym derivative</u> of ν with respect to μ and denote it $\frac{d\nu}{d\mu} := f$. □

Ex ① Let μ be a σ-finite signed measure on $(X,\mathcal{M})$. Then $\mu \ll |\mu|$. If $X = P \cup N$ is a Hahn decomposition for μ, then $\frac{d\mu}{d|\mu|} = 1_P - 1_N$:

$$\int_E \frac{d\mu}{d|\mu|}\, d|\mu| = \int_E 1_P - 1_N \, d|\mu| = |\mu|(E\cap P) - |\mu|(E \cap N) = \mu^+(E) - \mu^-(E) = \mu(E)$$

Similarly $\frac{d\mu^+}{d|\mu|} = 1_P$ and $\frac{d\mu^-}{d|\mu|} = 1_N$. Note that

$$\frac{d(\mu^+ - \mu^-)}{d|\mu|} = \frac{d\mu}{d|\mu|} = 1_P - 1_N = \frac{d\mu^+}{d|\mu|} - \frac{d\mu^-}{d|\mu|}$$

If ν is a σ-finite signed measure on $(X,\mathcal{M})$ with $\nu \ll \mu$, then $\nu \ll |\mu|$ and so we can consider $\frac{d\nu}{d|\mu|}$. If $\frac{d\nu}{d|\mu|} \in L^1(X,\mu)$ then $\frac{d\nu}{d|\mu|}(1_P - 1_N) \in L^1(X,\mu)$ with

$$\int_E \frac{d\nu}{d|\mu|}(1_P - 1_N)\, d\mu = \int_E \frac{d\nu}{d|\mu|}\, d|\mu| = \nu(E)$$

Thus we can define $\frac{d\nu}{d\mu} := \frac{d\nu}{d|\mu|}(1_P - 1_N)$.

② Let $(X,\mathcal{M},\mu)$ be a finite measure space, let $\mathcal{N} \subset \mathcal{M}$ be a σ-subalgebra, and define $\nu := \mu|_{\mathcal{N}}$. For real-valued $f \in L^1(X,\mu)$, define a finite signed measure λ on $(X,\mathcal{N})$ by:

$$\lambda(E) = \int_E f\, d\mu \qquad E \in \mathcal{N}.$$

Then $\lambda \ll \nu$ since if $0 = \nu(E) = \mu(E)$, then for any $F \subset E$ with $F \in \mathcal{N}$, $1_F f = 0$ μ-almost everywhere and so $\lambda(F) = 0$ by Proposition 2.16. Then $\mathbb{E}(f \mid \mathcal{N}) := \frac{d\lambda}{d\nu}$ is an $\mathcal{N}$-measurable function called the conditional expectation of f, and by Theorem 3.9 it is uniquely determined up to ν-almost everywhere equivalence by:

$$\int_E f\, d\mu = \int_E \mathbb{E}(f \mid \mathcal{N})\, d\nu \qquad E \in \mathcal{N}.$$

If $\mathcal{N} = \{\emptyset, X\}$, then $\mathbb{E}(f \mid \mathcal{N}) = \int_X f\, d\mu$ ($= \mathbb{E}(f)$ in probability notation).
If $\mathcal{N} = \mathcal{M}$, then $\mathbb{E}(f \mid \mathcal{M}) = f$.
Consider $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ for finite measures μ, ν. Then $\mathcal{M} \otimes \{\emptyset, Y\} \subset \mathcal{M} \otimes \mathcal{N}$ is a σ-subalgebra and for $f \in L^1(X \times Y, \mu \times \nu)$

$$\mathbb{E}(f \mid \mathcal{M} \otimes \{\emptyset, Y\}) = \int_Y f(\cdot, y)\, d\nu(y)$$

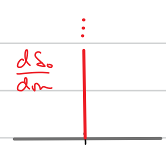
Indeed $E \in \mathcal{M} \otimes \{\emptyset, Y\}$ is either empty or of the form $E = F \otimes Y$ for $F \in \mathcal{M}$. Thus $1_E(x,y) = 1_F(x)$ and Fubini's theorem implies:

$$\int_E f\, d(\mu \times \nu) = \int_X \int_Y 1_E f\, d\nu\, d\mu = \int_X 1_F(x) \int_Y f(x,y)\, d\nu(y)\, d\mu(x) = \int_F \left[\int_Y f(x,y)\, d\nu(y)\right] d\mu(x).$$

Similarly $\mathbb{E}(f \mid \{\emptyset, X\} \otimes \mathcal{N}\} = \int_X f(x, \cdot)\, d\mu(x)$.

③ On $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ we have $\delta_0 \perp m$. In particular we do not have $\delta_0 \ll m$. However the Dirac delta function "$\frac{d\delta_0}{dm}$" is often used heuristically in physics as a function satisfying

$$\int_{\mathbb{R}} g(t) \frac{d\delta_0}{dm}(t)\, dt \text{ "="} g(0)$$



$\square$

As example ① (and the notation itself suggests) the Radon-Nikodym derivative shares many algebraic properties with the usual derivative:

$$\frac{d(\nu_1+\nu_2)}{d\mu} = \frac{d\nu_1}{d\mu} + \frac{d\nu_2}{d\mu}.$$

We even have a chain-rule:

Theorem 3.10 On $(X,\mathcal{M})$, let ν be a σ-finite signed measure and let μ, λ be σ-finite measures. Suppose $\nu \ll \mu$ and $\mu \ll \lambda$.

① $\nu \ll \lambda$ with
$$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$$

② For $g \in L^1(X,\nu)$, $g\frac{d\nu}{d\mu} \in L^1(X,\mu)$ with
$$\int_X g\frac{d\nu}{d\mu}\,d\mu = \int_X g\,d\nu.$$

Proof We first observe that ① follows from applying ② to μ and λ: for $E \in \mathcal{M}$

$$\int_E \frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}\,d\lambda = \int_X \left(1_E\frac{d\nu}{d\mu}\right)\frac{d\mu}{d\lambda}\,d\lambda = \int_X 1_E\frac{d\nu}{d\mu}\,d\mu = \nu(E) = \int_E \frac{d\nu}{d\lambda}\,d\lambda$$

Thus $\frac{d\mu}{d\nu}\frac{d\nu}{d\lambda} = \frac{d\mu}{d\lambda}$ λ-almost everywhere by Proposition 2.23.

To show ②, note that for $g \in L^+(X,\nu^\pm)$ we have

$$\int_X g\,\frac{d\nu^\pm}{d\mu}\,d\mu = \int_X g\,d\nu^\pm$$

by Exercise 3 on Homework 5. This then extends to $g \in L^1(X,\nu) = L^1(X,\nu^+)\cap L^1(X,\nu^-)$ by taking the appropriate complex combination of integrals of $\operatorname{Re}(g)^\pm$, $\operatorname{Im}(g)^\pm$. Finally, using $d\nu/d\mu = d\nu^+/d\mu - d\nu^-/d\mu$ we have

$$\int_X g\frac{d\nu}{d\mu}\,d\mu = \int_X g\frac{d\nu^+}{d\mu}\,d\mu - \int_X g\frac{d\nu^-}{d\mu}\,d\mu = \int_X g\,d\nu^+ - \int_X g\,d\nu^- = \int_X g\,d\nu$$

for $g \in L^1(X,\nu)$. □

Def For signed measures μ and ν on $(X,\mathcal{M})$, we say μ and ν are equivalent if $\mu \ll \nu$ and $\nu \ll \mu$. In this case we write $\mu \sim \nu$. □

For $\mu \sim \nu$, $E \in \mathcal{M}$ is μ-null iff it is ν-null. Thus μ-almost everywhere has the same meaning as ν-almost everywhere. We have already seen $\nu \sim |\nu|$.

Corollary 3.11 For σ-finite measures μ and ν on $(X,\mathcal{M})$, if $\mu\sim\nu$ then $\frac{d\mu}{d\nu}\frac{d\nu}{d\mu} = 1$ μ-a.e. (equivalently ν-a.e.)

Proof Since $\frac{d\mu}{d\mu} = 1$ μ-almost everywhere, this is immediate from Theorem 3.10.① □

Proposition 3.12 For measures $\mu_1,\ldots,\mu_n$ on $(X,\mathcal{M})$, $\mu := \sum_{j=1}^n \mu_j$ satisfies $\mu_j \ll \mu$ for $j=1,\ldots,n$.

Proof This follows from $0 \le \mu_j(E) \le \mu(E)$ for all $E\in\mathcal{M}$. □

3.3 Complex Measures

Def A <u>complex measure</u> on a measurable space $(X,\mathcal{M})$ is a map $\nu:\mathcal{M}\to\mathbb{C}$ such that

1. $\nu(\phi)=0$
2. If $(E_n)_{n\in\mathbb{N}}\subset\mathcal{M}$ are disjoint then
$$\nu\left(\bigcup_{n=1}^{\infty}E_n\right)=\sum_{n=1}^{\infty}\nu(E_n)$$
where the series converges absolutely. □

The signed measures defined by $\nu_r(E):=\mathrm{Re}(\nu(E))$ and $\nu_i(E):=\mathrm{Im}(\nu(E))$ are called the <u>real</u> and <u>imaginary parts</u> of ν, respectively. We define
$$L^1(X,\nu):=L^1(X,\nu_r)\cap L^1(X,\nu_i)$$
and for $f\in L^1(X,\nu)$ we define its integral with respect to ν as
$$\int_X f\,d\nu:=\int_X f\,d\nu_r+i\int_X f\,d\nu_i$$
□

Note that a complex measure does <u>not</u> take any infinite values. Hence ν_r and ν_i are also finite and $\nu(\mathcal{M})$ is a <u>bounded</u> subset of $\mathbb{C}$ (Exercise check this).

Ex Let $(X,\mathcal{M},\mu)$ be a measure space. For any $f\in L^1(X,\mu)$,
$$\nu(E):=\int_E f\,d\mu$$
defines a complex measure since for $(E_n)_{n\in\mathbb{N}}\subset\mathcal{M}$ the monotone convergence theorem implies
$$\sum_{n=1}^{\infty}|\nu(E_n)|\le\sum_{n=1}^{\infty}\int_{E_n}|f|\,d\mu=\int_{\cup E_n}|f|\,d\mu<\infty$$
and so the dominated convergence implies
$$\sum_{n=1}^{\infty}\nu(E_n)=\sum_{n=1}^{\infty}\int_{E_n}f\,d\mu=\int_{\bigcup_{n=1}^{\infty}E_n}f\,d\mu=\nu\left(\bigcup_{n=1}^{\infty}E_n\right).$$
The real and imaginary parts are:
$$\nu_r(E)=\int_E \mathrm{Re}(f)\,d\mu \qquad\text{and}\qquad \nu_i(E)=\int_E \mathrm{Im}(f)\,d\mu.$$
Thus $\nu_r,\nu_i\ll\mu$ with $\frac{d\nu_r}{d\mu}=\mathrm{Re}(f)$ and $\frac{d\nu_i}{d\mu}=\mathrm{Im}(f)$, so by Theorem 3.10.2 we have for $g\in L^1(X,\nu)$
$$\int_X g\,d\nu=\int_X g\,d\nu_r+i\int_X g\,d\nu_i=\int_X g\,\mathrm{Re}(f)\,d\mu+i\int_X g\,\mathrm{Im}(f)\,d\mu=\int_X g f\,d\mu.$$
□

We write $d\nu=f\,d\mu$ for the measure defined in the previous example. Note that $\mu(E)=0$ implies $\nu(E)=0$.

Def For a complex measure ν on $(X,\mathcal{M})$, we say $E\in\mathcal{M}$ is a <u>ν-null set</u> if $\nu(F)=0$ for all $F\subset E$ with $F\in\mathcal{M}$. If μ is a complex (or signed) measure on $(X,\mathcal{M})$, we say ν and

μ are mutually singular and write $\nu \perp \mu$ if there exists a partition $X = E \cup F$ so that E is ν-null and F is μ-null. We say ν is absolutely continuous with respect to μ and write $\nu \ll \mu$ if every μ-null set is also a ν-null set. We say ν and μ are equivalent and write $\nu \sim \mu$ if $\nu \ll \mu$ and $\mu \ll \nu$. □

For a complex measure ν on $(X, \mathcal{M})$, $E \in \mathcal{M}$ is ν-null iff E is ν_r-null and ν_i-null. Consequently

$$\nu \perp \mu \iff \nu_r \perp \mu \text{ and } \nu_i \perp \mu$$
$$\nu \ll \mu \iff \nu_r \ll \mu \text{ and } \nu_i \ll \mu$$

Theorem 3.13 (The (complex) Lebesgue–Radon–Nikodym Theorem) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For a complex measure ν on $(X, \mathcal{M})$, there exists unique complex measures λ, ρ on $(X, \mathcal{M})$ satisfying

$$\nu = \lambda + \rho$$

where $\lambda \perp \mu$ and $\rho \ll \mu$. Moreover, there exists $f \in L^1(X, \mu)$ such that $d\rho = f\, d\mu$, and f is unique up to μ-almost everywhere equivalence.

Proof This follows from applying Theorem 3.9 to ν_r and ν_i separately. That f is integrable follows from ρ (and hence ρ_r, ρ_i) being finite. □

Def Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. For ν a complex measure on $(X, \mathcal{M})$ satisfying $\nu \ll \mu$, the function f such that $d\nu = f\, d\mu$ is called the Radon–Nikodym derivative of ν with respect to μ, and is denoted $\frac{d\nu}{d\mu} := f$. □

Our computation in the above example therefore gives:

Corollary 3.14 Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space. If ν is a signed measure on $(X, \mathcal{M})$ satisfying $\nu \ll \mu$, then for $g \in L^1(X, \nu)$ one has $g \frac{d\nu}{d\mu} \in L^1(X, \mu)$ with

$$\int_X g\, d\nu = \int_X g \frac{d\nu}{d\mu}\, d\mu.$$

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Given a complex measure ν on $(X, \mathcal{M})$, we wish to define its total variation $|\nu|$. In the signed measure case, this was easy to do because the positive and negative variations were mutually singular. But ν_r and ν_i are not mutually singular, and so it is not clear that the naive formula

$$|\nu|(E) \stackrel{?}{=} \left(|\nu_r|(E)^2 + |\nu_i|(E)^2 \right)^{1/2}$$

even defines a measure. However, the Radon–Nikodym allows us to play with formulas like this for functions. In particular, recall from Exercise 3.(a) on Homework 8 that if ν is signed measure and $\nu \ll \mu$ for a σ-finite measure, then $\frac{d|\nu|}{d\mu} = \left|\frac{d\nu}{d\mu}\right|$. In the complex measure case, we always have $\nu \ll |\nu_r| + |\nu_i|$ (check this) and $|\nu_r| + |\nu_i|$ is finite.

Def For a complex measure ν on $(X,\mathcal{M})$, the total variation of ν is the measure

$$d|\nu| := \left|\frac{d\nu}{d(|\nu_r|+|\nu_i|)}\right| d(|\nu_r|+|\nu_i|)$$

We denote by $M(X,\mathcal{M})$ the $\mathbb{C}$-vector space of all complex measures on $(X,\mathcal{M})$, and for $\nu \in M(X,\mathcal{M})$ the total variation norm of ν is $\|\nu\| := |\nu|(X)$. □

By Exercise 6 on Homework 8, the total variation norm is indeed a norm, and moreover $M(X,\mathcal{M})$ is a Banach space with this norm: it is complete with respect to the metric $\|\nu-\mu\|$.

Proposition 3.15 Let ν be a signed measure on $(X,\mathcal{M})$. For any σ-finite measure μ on $(X,\mathcal{M})$ satisfying $\nu \ll \mu$, one has

$$d|\nu| = \left|\frac{d\nu}{d\mu}\right| d\mu.$$

Proof Let $\lambda := |\nu_r|+|\nu_i|$ (so that $d|\nu| = |d\nu/d\lambda|\, d\lambda$ by definition) and let $\rho := \lambda + \mu$. Then we have $\nu \ll \mu, \lambda \ll \rho$ and so by Theorem 3.10.① we have

$$\frac{d\nu}{d\mu}\frac{d\mu}{d\rho} = \frac{d\nu}{d\rho} = \frac{d\nu}{d\lambda}\frac{d\lambda}{d\rho}$$

ρ-almost everywhere. Therefore

$$\left|\frac{d\nu}{d\mu}\right| \frac{d\mu}{d\rho} = \left|\frac{d\nu}{d\mu}\frac{d\mu}{d\rho}\right| = \left|\frac{d\nu}{d\lambda}\frac{d\lambda}{d\rho}\right| = \left|\frac{d\nu}{d\lambda}\right|\frac{d\lambda}{d\rho}$$

ρ-almost everywhere and so

$$d|\nu| = \left|\frac{d\nu}{d\lambda}\right| d\lambda = \left|\frac{d\nu}{d\lambda}\right|\frac{d\lambda}{d\rho}\, d\rho = \left|\frac{d\nu}{d\mu}\right|\frac{d\mu}{d\rho}\, d\rho = \left|\frac{d\nu}{d\mu}\right| d\mu.$$ □

Proposition 3.16 Let ν be a complex measure on $(X,\mathcal{M})$.

① $|\nu(E)| \leq |\nu|(E)$ for all $E \in \mathcal{M}$

② $\nu \sim |\nu|$ and $\left|\frac{d\nu}{d|\nu|}\right| = 1$ $|\nu|$-a.e.

③ $L^1(X,\nu) = L^1(X,|\nu|)$, and for $f \in L^1(X,\nu)$

$$\left|\int_X f\, d\nu\right| \leq \int_X |f|\, d|\nu|$$

Proof ①: Let $\mu := |\nu_r|+|\nu_i|$. Then by Proposition 2.22 we have

$$|\nu(E)| = \left|\int_E \frac{d\nu}{d\mu}\, d\mu\right| \leq \int_E \left|\frac{d\nu}{d\mu}\right| d\mu = |\nu|(E)$$

②: The previous part shows $\nu \ll |\nu|$, and $|\nu| \ll \mu \ll \nu$ implies $\nu \sim |\nu|$. Observe that for $E \in \mathcal{M}$

$$\int_E \frac{d\nu}{d\mu}\, d\mu = \nu(E) = \int_E \frac{d\nu}{d|\nu|}\, d|\nu| = \int_E \frac{d\nu}{d|\nu|}\left|\frac{d\nu}{d\mu}\right| d\mu,$$

and so $\frac{d\nu}{d\mu} = \frac{d\nu}{d|\nu|}\left|\frac{d\nu}{d\mu}\right|$ μ-almost everywhere (and hence $|\nu|$-a.e since $\mu \sim \nu \sim |\nu|$). Observe that if $E := \{x \in X : |\frac{d\nu}{d\mu}(x)| = 0\}$, then

$$|\nu|(E) = \int_E \left|\frac{d\nu}{d\mu}\right| d\mu = \int_E 0\, d\mu = 0$$

Thus $\left|\frac{d\nu}{d\mu}\right| > 0$ $|\nu|$-a.e., and so $\left|\frac{d\nu}{d\mu}\right| = \left|\frac{d\nu}{d|\nu|}\right| \left|\frac{d\nu}{d\mu}\right|$ $|\nu|$-a.e. implies $\left|\frac{d\nu}{d|\nu|}\right| = 1$ $|\nu|$-a.e.

3: Part 1 implies $|\nu_r| \le |\nu|$ and $|\nu_i| \le |\nu|$. Hence by Exercise 5.(a) on Homework 7

$$L^1(X,|\nu|) \subset L^1(X,|\nu_r|) \cap L^1(X,|\nu_i|) = L^1(X,\nu_r) \cap L^1(X,\nu_i) = L^1(X,\nu)$$

Conversely, Exercise 3.(a) on Homework 8 gives:

$$|\nu|(E) = \int_E \left|\frac{d\nu}{d|\nu|}\right| d|\nu| \le \int_E \left|\frac{d\nu_r}{d|\nu|}\right| + \left|\frac{d\nu_i}{d|\nu|}\right| d|\nu| = \int_E \frac{d|\nu_r|}{d|\nu|} + \frac{d|\nu_i|}{d|\nu|} d|\nu| = |\nu_r|(E) + |\nu_i|(E)$$

So $|\nu| \le |\nu_r| + |\nu_i|$, which implies

$$L^1(X,\nu) = L^1(X,|\nu_r|) \cap L^1(X,|\nu_i|) \subset L^1(X,|\nu|).$$

Hence $L^1(X,\nu) = L^1(X,|\nu|)$.

Now, for a simple function with standard representation $\phi = \sum_{j=1}^{n} \alpha_j 1_{E_j}$, we have by 1 that:

$$\left|\int_X \phi \, d\nu\right| = \left|\sum_{j=1}^{n} \alpha_j \nu(E_j)\right| \le \sum_{j=1}^{n} |\alpha_j| \, |\nu|(E_j) = \int_X |\phi| \, d|\nu|.$$

The general inequality then follows by approximating $f \in L^1(X,\nu)$ by simple functions and using the dominated convergence theorem (Theorem 2.24). □

Since $\nu \sim |\nu|$ by Part 2 of the above theorem, we have the following characterizations for mutual singularity, absolute continuity and equivalence of complex measures ν, μ:

$$\nu \perp \mu \iff |\nu| \perp |\mu|$$
$$\nu \ll \mu \iff |\nu| \ll |\mu|$$
$$\nu \sim \mu \iff |\nu| \sim |\mu|.$$

Proposition 3.17 If ν, μ are complex measures on $(X, \mathcal{M})$, then $|\nu+\mu| \le |\nu| + |\mu|$.

Proof Since $\nu \sim |\nu| \le |\nu| + |\mu|$ we have $\nu \ll |\nu|+|\mu|$ and so by Proposition 3.15

$$d|\nu| = \left|\frac{d\nu}{d(|\nu|+|\mu|)}\right| d(|\nu|+|\mu|).$$

Similarly for $|\mu|$. Thus for $E \in \mathcal{M}$

$$|\nu+\mu|(E) = \int_E \left|\frac{d(\nu+\mu)}{d(|\nu|+|\mu|)}\right| d(|\nu|+|\mu|)$$
$$\le \int_E \left|\frac{d\nu}{d(|\nu|+|\mu|)}\right| + \left|\frac{d\mu}{d(|\nu|+|\mu|)}\right| d(|\nu|+|\mu|) = |\nu|(E) + |\mu|(E)$$

□

3.4 Differentiation on Euclidean Space

Recall that the fundamental theorem of calculus (or at least one of its parts) says that if $f:[a,b]\to\mathbb{R}$ is continuous, then $F:[a,b]\to\mathbb{R}$ defined by

$$F(x):=\int_a^x f(t)\,dt$$

is differentiable on (a,b) with $F'=f$. Equivalently, for each $x\in(a,b)$

$$\frac{1}{m(B(x,\varepsilon))}\int_{B(x,\varepsilon)} f(t)\,dt = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} f(t)\,dt = \frac{F(x+\varepsilon)-F(x-\varepsilon)}{2\varepsilon} \xrightarrow{\varepsilon\to 0} F'(x)=f(x).$$

Our main goal of this section is to use the Lebesgue measure on $\mathbb{R}^n$ to show this limit holds for a much ~~broader~~ class of functions.

Throughout, we fix $n\in\mathbb{N}$ and consider the measure space $(\mathbb{R}^n,\mathcal{L}^n,m^n)$. We will typically write m for m^n, and "measurable", "integrable", and "almost everywhere" will mean with respect to this measure space. We first require a technical geometric lemma:

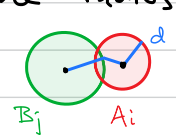
Lemma 3.18 Let $\mathcal{C}$ be a collection of open balls in $\mathbb{R}^n$, and denote

$$\mathcal{U}:=\bigcup_{B\in\mathcal{C}} B$$

If $m(\mathcal{U})>c>0$, then there exists disjoint $B_1,\ldots,B_k\in\mathcal{C}$ such that

$$m(B_1)+\cdots+m(B_k)>3^{-n}c.$$

Proof Using Theorem 2.38 we can find $K\subset\mathcal{U}$ compact with $m(K)>c$. Since $\mathcal{C}$ covers K, the compactness implies we can find a finite subcover, say by $A_1,\ldots,A_m\in\mathcal{C}$. Let B_1 be the A_i with largest radius, let B_2 be the A_i with the largest radius that is disjoint from B_1, let B_3 be the A_i with the largest radius that is disjoint from B_1 and B_2, and so on until there are no A_i's disjoint from $B_1,B_2,\ldots,B_d$. Now if A_i does not appear in the list $B_1,\ldots,B_d$ then, $A_i\cap B_j\neq\emptyset$ for some j. Moreover, if j is the smallest index with this property (so that $A_i\cap B_{j-1}=\emptyset$) then the radius of A_i must be less than or equal to the radius of B_j (by definition of B_j). Consequently, if B_j^* is the ball with radius three times that of B_j but with the same center, then $A_i\subset B_j^*$. So



$d<\mathrm{rad}(B_j)+2\,\mathrm{rad}(A_i)\le 3\,\mathrm{rad}(B_j)$

$$K\subset A_1\cup\cdots\cup A_m\subset B_1^*\cup\cdots\cup B_d^*$$

and therefore by Theorem 2.40

$$c<m(K)\le\sum_{j=1}^d m(B_j^*)=\sum_{j=1}^d 3^n\,m(B_j).$$

$\square$

Def We call an $\mathcal{L}^n$-measurable function $f: \mathbb{R}^n \to \mathbb{C}$ locally integrable with respect to m if

$$\int_K |f| \, dm < \infty$$

for all bounded measurable sets $K \subset \mathbb{R}^n$. We denote the set of such functions by $L^1_{loc}(\mathbb{R}^n, m)$. Given $f \in L^1_{loc}(\mathbb{R}^n, m)$, $x \in \mathbb{R}^n$, and $r > 0$ we denote

$$A_r f(x) := \frac{1}{m(B(x,r))} \int_{B(x,r)} f \, dm$$

and call this quantity the average of f over $B(x,r)$. □

Lemma 3.19 For $f \in L^1_{loc}(\mathbb{R}^n, m)$, the map

$$\mathbb{R}^n \times (0,\infty) \ni (x,r) \longmapsto A_r f(x)$$

is continuous.

Proof If $(x_n, r_n) \to (x_0, r_0)$ then $1_{B(x_n,r_n)} \to 1_{B(x_0,r_0)}$ pointwise on $\mathbb{R}^n \setminus S(x_0,r_0)$ where $S(x_0,r_0) = \{y \in \mathbb{R}^n : |x-y| = r\}$. Since we showed $m(S(x_0,r_0)) = 0$ at the end of Chapter 2, we see that $1_{B(x_n,r_n)} f \to 1_{B(x_0,r_0)} f$ m-almost everywhere. By dropping sufficiently many terms at the beginning of our sequence, we may assume

$$|x_n - x_0| < \tfrac{1}{2} \quad \text{and} \quad |r_n - r_0| < \tfrac{1}{2} \quad \forall n \in \mathbb{N}.$$

Hence $B(x_n,r_n) \subset B(x_0, r_0+1)$ for all $n \in \mathbb{N}$, and so $|f 1_{B(x_n,r_n)}| \leq |f| 1_{B(x_0,r_0+1)}$. The latter function is integrable since f is locally integrable, and so the dominated convergence theorem (Theorem 2.24) implies

$$\lim_{n\to\infty} \int_{B(x_n,r_n)} f \, dm = \int_{B(x_0,r_0)} f \, dm.$$

Thus $(x,r) \mapsto \int_{B(x,r)} f \, dm$ is continuous. Also $\frac{1}{m(B(x,r))} = \frac{1}{c} r^{-n}$ for some $c > 0$ by the example at the end of Chapter 2. Thus

$$A_r f(x) = \frac{1}{c} r^{-n} \int_{B(x,r)} f \, dm$$

is continuous as the product of continuous functions. □

Def For $f \in L^1_{loc}(\mathbb{R}^n, m)$, its Hardy-Littlewood maximal function $Hf: \mathbb{R}^n \to [0,\infty]$ is defined by

$$Hf(x) := \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f| \, dm$$

□

Observe that Hf is always $\mathcal{L}^n$-measurable since

$$(Hf)^{-1}((a,\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((a,\infty))$$

(if $Hf(x) > a$, then $A_r|f|(x) > a$ for some $r > 0$), and $(A_r|f|)^{-1}((a,\infty))$ are open by Lemma 3.19.

Theorem 3.20 (The Hardy-Littlewood Maximal Inequality) There exists $C>0$ such that for all $f\in L^1(\mathbb{R}^n, m)$ and all $\varepsilon>0$

$$m(\{x\in X : Hf(x)>\varepsilon\}) \le \frac{C}{\varepsilon}\int_{\mathbb{R}^n}|f|\,dm$$

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Proof Let $E:=\{x\in X : Hf(x)>\varepsilon\}$. Then for each $x\in E$ there exists $r(x)>0$ so that $A_{r(x)}|f|>\varepsilon$, or equivalently

$$m(B(x,r(x))) < \frac{1}{\varepsilon}\int_{B(x,r(x))}|f|\,dm. \quad (*)$$

Now, since

$$E\subset \bigcup_{x\in E} B(x,r(x)),$$

if $m(E)>c$ for some $c>0$, then by Lemma 3.18 there exists $x_1,\ldots,x_d\in E$ so that the $B(x_j,r(x_j))$ are disjoint and

$$\sum_{j=1}^d m(B(x_j,r(x_j)))>3^{-n}c.$$

Combining this with $(*)$ gives:

$$c<3^n\sum_{j=1}^d m(B(x_j,r(x_j)))\le \frac{3^n}{\varepsilon}\sum_{j=1}^d\int_{B(x_j,r(x_j))}|f|\,dm \le \frac{3^n}{\varepsilon}\int_{\mathbb{R}^n}|f|\,dm.$$

Letting $c\nearrow m(E)$ yields the claimed inequality. $\square$

Recall that for a metric space (X,d) and a function $f:X\to\mathbb{R}$, one defines the limit inferior/superior for f at $x_0\in X$ by:

$$\liminf_{x\to x_0} f(x) = \lim_{\varepsilon\to 0}\inf_{d(x,x_0)<\varepsilon} f(x) \quad \text{and} \quad \limsup_{x\to x_0} f(x) = \lim_{\varepsilon\to 0}\sup_{d(x,x_0)<\varepsilon} f(x)$$

Exercise Show $\lim_{x\to x_0} f(x)$ exists iff $\limsup_{x\to x_0} f(x) \le \liminf_{x\to x_0} f(x)$ iff there exists $c\in\mathbb{R}$ so that $\limsup_{x\to x_0}|f(x)-c|=0$, in which case $\lim_{x\to x_0} f(x)=c$.

Note in particular that for $f\in L^1_{loc}(\mathbb{R}^n,m)$ and $x\in\mathbb{R}^n$

$$\limsup_{r\to 0}|A_r f(x)| \le \lim_{\varepsilon\to 0}\sup_{0<r<\varepsilon} A_r|f|(x) \le Hf(x)$$

Theorem 3.21 For $f\in L^1_{loc}(\mathbb{R}^n,m)$,

$$f(x) = \lim_{r\to 0} A_r f(x) = \lim_{r\to 0}\frac{1}{m(B(x,r))}\int_{B(x,r)} f\,dm$$

for m-almost every $x\in\mathbb{R}^n$.

Proof We first note that it suffices to prove this for functions in $L^1(\mathbb{R}^n,m)$. Indeed, for $f\in L^1_{loc}(\mathbb{R}^n,m)$, if $N\in\mathbb{N}$, then for $r\le 1$ and m-almost every $x\in B(0,N)$

$$A_r f(x) = A_r(f 1_{B(0,N+1)})(x) \xrightarrow{r\to 0} (f 1_{B(0,N+1)})(x) = f(x).$$

since $f 1_{B(0,N+1)}\in L^1(\mathbb{R}^n,m)$. Thus

$$\{x \in \mathbb{R}^n : A_r f(x) \not\to f(x)\} = \bigcup_{N=1}^{\infty} \{x \in B(0,N) : A_r f(x) \not\to f(x)\}$$

is an m-null set.

So let $f \in L^1(\mathbb{R}^n, m)$ and let $\varepsilon > 0$. Using Theorem 2.39 we can find a continuous function $g : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\int_{\mathbb{R}^n} |f - g| \, dm < \varepsilon.$$

Given $x \in \mathbb{R}^n$, the continuity of g implies there exists $\delta > 0$ so that $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Hence for $r \leq \delta$

$$|A_r g(x) - g(x)| \leq \frac{1}{m(B(x,r))} \int_{B(x,r)} |g(y) - g(x)| \, dm(y) \leq \varepsilon.$$

Thus $A_r g(x) \to g(x)$ as $r \to 0$ for every $x \in \mathbb{R}^n$. So we have

$$\limsup_{r \to 0} |A_r f(x) - f(x)| = \limsup_{r \to 0} |A_r(f-g)(x) + (A_r g - g)(x) + (g-f)(x)|$$

$$\leq H(f-g)(x) + 0 + |f(x) - g(x)|.$$

For $\alpha > 0$, denote

$$E_\alpha := \{x \in \mathbb{R}^n : \limsup_{r \to 0} |A_r f(x) - f(x)| \geq \alpha\} \quad \text{and} \quad F_\alpha := \{x \in \mathbb{R}^n : |f(x) - g(x)| \geq \alpha\}.$$

Then the above estimate implies

$$E_\alpha \subset F_{\alpha/2} \cup \{x \in \mathbb{R}^n : H(f-g)(x) \geq \tfrac{\alpha}{2}\}.$$

Using

$$m(F_{\alpha/2}) \leq \frac{2}{\alpha} \int_{F_{\alpha/2}} |f-g| \, dm \leq \frac{2\varepsilon}{\alpha}$$

and the Hardy-Littlewood maximal inequality (Theorem 3.20), we have

$$m(E_\alpha) \leq \frac{2\varepsilon}{\alpha} + \frac{2C\varepsilon}{\alpha}$$

for some $C > 0$. Since $\varepsilon > 0$ was arbitrary, we have $m(E_\alpha) = 0$ for all $\alpha > 0$. Hence

$$\{x \in \mathbb{R}^n : \limsup_{r \to 0} |A_r f(x) - f(x)| > 0\} = \bigcup_{n=1}^{\infty} E_{1/n}$$

is m-null, and so $A_r f(x) \to f(x)$ m-almost everywhere. $\square$

Since $f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) \, dm(y)$, the previous theorem is equivalent to

$$\lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(x) - f(y) \, dm(y) = 0$$

for m-almost every $x \in \mathbb{R}^n$. We will strengthen this in Theorem 3.22 below by replacing the integrand with its absolute value. We first introduce some terminology.

Def The Lebesgue set of $f \in L^1_{loc}(\mathbb{R}^n, m)$ is the set

$$L_f := \left\{x \in \mathbb{R}^n : \lim_{r \to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) = 0\right\}$$ $\square$

Theorem 3.22 For $f \in L^1_{loc}(\mathbb{R}^n, m)$, $m(L_f^c) = 0$.

Proof For each $z \in \mathbb{C}$, Theorem 3.21 applied to $g_z(x) := |f(x) - z| \in L^1_{loc}(\mathbb{R}^n, m)$ $\swarrow \notin L^1(\mathbb{R}^n, m)$ shows that

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - z| \, dm(y) \xrightarrow{r \to 0} |f(x) - z|$$

except for x in an m-null set $E_z \subset \mathbb{R}^n$. Then

$$E := \bigcup_{z \in \mathbb{Q} + i\mathbb{Q}} E_z$$

is m-null since $\mathbb{Q} + i\mathbb{Q}$ is countable. For $x \notin E$ and $\varepsilon > 0$, we can find $z \in \mathbb{Q} + i\mathbb{Q}$ so that $|f(x) - z| < \varepsilon$ and hence

$$\frac{1}{m(B(x,r))} \int_{B(x,r)} |f(x) - f(y)| \, dm(y) \le |f(x) - z| + \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - z| \, dm(y) \longrightarrow 2|f(x) - z| < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $x \in L_f$. Thus $E^c \subset L_f$ and so $m(L_f^c) \le m(E) = 0$. □

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We continue to strengthen this result by replacing $B(x,r)$ with a more generic class of sets:

Def We say a family of Borel subsets $\{E_r \subset \mathbb{R}^n : r > 0\}$ shrinks nicely to $x \in \mathbb{R}^n$ if $E_r \subset B(x,r)$ for all $r > 0$ and

$$\inf_{r > 0} \frac{m(E_r)}{m(B(x,r))} > 0.$$ □

Note that if $\alpha > 0$ is the infimum in the above definition, then $\alpha m(B(x,r)) \le m(E_r) \le m(B(x,r))$ for all $r > 0$.

Ex Let $U \subset B(0,1)$ be any Borel subset with $m(U) > 0$. For $x \in \mathbb{R}^n$ and $r > 0$, if

$$E_r := \{x + ry : y \in U\} = U \cdot r + x$$

then $\{E_r : r > 0\}$ shrinks nicely to x with

$$\inf_{r > 0} \frac{m(E_r)}{B(x,r)} = \frac{m(U)}{m(B(0,1))} > 0.$$

Note that if $0 \notin U$ then $x \notin E_r$ for any $r > 0$. □

Theorem 3.23 (The Lebesgue Differentiation Theorem)
Let $f \in L^1_{loc}(\mathbb{R}^n, m)$. For every $x \in L_f$ the Lebesgue set of f — in particular, for almost every $x \in \mathbb{R}^n$ — we have

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(x) - f(y)| \, dm(y) = 0$$

and

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_E f \, dm = f(x)$$

for every family $\{E_r : r > 0\}$ that shrinks nicely to x.

Proof Denote

$$\alpha := \inf_{r > 0} \frac{m(E_r)}{m(B(x,r))}$$

so that $m(E_r) \geq \alpha\, m(B(x,r))$ for all $r>0$. We have for $x \in L_f$ that

$$\frac{1}{m(E_r)}\int_{E_r} |f(x)-f(y)|\,dm(y) \leq \frac{1}{m(E_r)}\int_{B(x,r)} |f(x)-f(y)|\,dm(y) \leq \frac{1}{\alpha\, m(B(x,r))}\int_{B(x,r)} |f(x)-f(y)|\,dm(y) \longrightarrow 0.$$

This proves the first limit. The second limit follows from

$$\left| f(x) - \frac{1}{m(E_r)}\int_{E_r} f(y)\,dm(y)\right| = \left| \frac{1}{m(E_r)}\int_{E_r} f(x)-f(y)\,dm(y)\right| \leq \frac{1}{m(E_r)}\int_{E_r} |f(x)-f(y)|\,dm(y),$$

and the first limit. □

We can interpret the above results in terms of measures: suppose $\nu \in M(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ with $\nu \ll m$ and $\frac{d\nu}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$. Then Theorem 3.23 implies

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = \frac{d\nu}{dm}(x)$$

for any family $\{E_r : r>0\}$ that shrinks nicely to a point $x \in L_{\frac{d\nu}{dm}}$. Let us generalize this a bit more.

Def We call a Borel measure ν on $\mathbb{R}^n$ <u>regular</u> if

1. $\nu(K) < \infty$ for every compact $K \subset \mathbb{R}^n$;
2. for every $E \in \mathcal{B}_{\mathbb{R}^n}$

$$\nu(E) = \inf\{\nu(U) : U \supset E \text{ open}\}$$

A signed or complex Borel measure ν is <u>regular</u> if $|\nu|$ is regular in the above sense. □

Note that 1 implies any regular signed or complex Borel measure is σ-finite. Consequently one can consider the Lebesgue decomposition

$$\nu = \lambda + \rho$$

of ν with respect to m: $\lambda \perp m$ and $\rho \ll m$.

Theorem 3.24 Let ν be a regular signed or complex Borel measure on $\mathbb{R}^n$ with Lebesgue decomposition $\nu = \lambda + \rho$ with respect to m, where $\lambda \perp m$ and $\rho \ll m$. Then for m-almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{\nu(E_r)}{m(E_r)} = \frac{d\rho}{dm}(x)$$

for every family $\{E_r : r>0\}$ that shrinks nicely to x.

Before proving this theorem, we first require a lemma:

Lemma 3.25 Let ρ be a signed or complex Borel measure on $\mathbb{R}^n$ satisfying $\rho \ll m$. Then ρ is regular iff $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$.

Proof Recall that in both the signed measure and complex measure cases $\left|\frac{d\rho}{dm}\right| = \frac{d|\rho|}{dm}$ (by Exercise 3.(a) on Homework 8 and by Proposition 3.15, respectively). Thus $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$

iff

$$|\rho|(K) = \int_K \frac{d|\rho|}{dm}\, dm = \int_K \left|\frac{d\rho}{dm}\right| dm < \infty.$$

for all bounded $K \in \mathcal{B}_{\mathbb{R}^n}$. Since every bounded $K \subset \mathbb{R}^n$ is contained in a compact set (namely its closure) this is further equivalent to $|\rho|(K) < \infty$ for all compact $K \subset \mathbb{R}^n$. So it remains to show that $\frac{d\rho}{dm} \in L^1_{loc}(\mathbb{R}^n, m)$ implies

$$|\rho|(E) = \inf\{|\rho|(U) : U \supset E \text{ open}\} \qquad *$$

for all $E \in \mathcal{B}_{\mathbb{R}^n}$. First suppose E is bounded, say $E \subset B(0,R)$ for $R > 0$. Note that $f := \left|\frac{d\rho}{dm}\right| 1_{B(0,R)} \in L^1(\mathbb{R}^n, m)$, and so by Corollary 3.6 given $\varepsilon > 0$ there exists $\delta > 0$ so that $m(A) < \delta$ implies

$$\int_A f\, dm < \varepsilon.$$

By Theorem 2.38, $\exists\, U \supset E$ open so that $m(U) < m(E) + \delta$, and hence $m(U \setminus E) < \delta$ since $m(E) < \infty$ by virtue of E being bounded. By considering $U \cap B(0,R)$, we can assume $U \setminus E \subset B(0,R)$. Thus

$$|\rho|(U) = |\rho|(E) + \int_{U/E} \left|\frac{d\rho}{dm}\right| dm = |\rho|(E) + \int_{U/E} f\, dm < |\rho|(E) + \varepsilon,$$

and $(*)$ follows.

For potentially unbounded $E \in \mathcal{B}_{\mathbb{R}^n}$, let $E_n := E \cap B(0,n)$. Using the above we can find $U_n \supset E_n$ open such that $|\rho|(U_n \setminus E_n) < \frac{1}{2^n}\varepsilon$ for some $\varepsilon > 0$. Then

$$U := \bigcup_{n=1}^{\infty} U_n$$

is open, contains E, and satisfies $|\rho|(U \setminus E) < \varepsilon$. Thus $|\rho|(U) < |\rho|(E) + \varepsilon$. $\square$

Remark If $\rho \in M(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n})$ satisfies $\rho \ll m$, then $\frac{d\rho}{dm} \in L^1(\mathbb{R}^n, m)$ since ρ is finite. Thus the previous proposition implies every complex Borel measure on $\mathbb{R}^n$ that is absolutely continuous with respect to the Lebesgue measure m is regular. $\square$

Proof (Theorem 3.24) By Homework 9, λ and ρ are both regular. Lemma 3.25 and the Lebesgue differentiation theorem (Theorem 3.23) imply for m-almost every $x \in \mathbb{R}^n$

$$\lim_{r\to 0} \frac{\rho(E_r)}{m(E_r)} = \lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} \frac{d\rho}{dm}\, dm = \frac{d\rho}{dm}(x)$$

for every family $\{E_r : r > 0\}$ that shrinks nicely to x. Thus it suffices to show $\frac{\lambda(E_r)}{m(E_r)} \to 0$ as $r \to 0$. Let

$$\alpha := \inf_{r>0} \frac{m(E_r)}{m(B(x,r))}.$$

Then

$$\left|\frac{\lambda(E_r)}{m(E_r)}\right| \le \frac{|\lambda|(E_r)}{m(E_r)} \le \frac{|\lambda|(B(x,r))}{m(E_r)} \le \frac{|\lambda|(B(x,r))}{\alpha\, m(B(x,r))}.$$

So it suffices to show $|\lambda|(B(x,r))/m(B(x,r)) \longrightarrow 0$ as $r \to 0$. Recall $\lambda \perp m$ implies $|\lambda| \perp m$, and so there exists a partition $\mathbb{R}^n = A \cup B$ so that A is $|\lambda|$-null and B is m-null. Define

$$F_k := \left\{x \in A : \limsup_{r\to 0} \frac{|\lambda|(B(x,r))}{m(B(x,r))} > \frac{1}{k}\right\},$$

so that showing $m(F_k)=0$ for all $k\in\mathbb{N}$ will complete the proof. Given $\varepsilon>0$, the regularity of $|\lambda|$ implies there exists $U_\varepsilon\supset A$ open such that $|\lambda|(U_\varepsilon)<\varepsilon$. For each $x\in F_k$, there exists $r(x)$ so that $B(x,r(x))\subset U_\varepsilon$ and $|\lambda|(B(x,r(x)))>\frac{1}{k}m(B(x,r(x)))$. Let

$$V_\varepsilon := \bigcup_{x\in F_k} B(x,r(x)),$$

which is contained in U_ε. If $m(V_\varepsilon)>c>0$, then Lemma 3.18 implies there exists $x_1,\ldots,x_d\in F_k$ so that $B(x_1,r(x_1)),\ldots,B(x_d,r(x_d))$ are disjoint and

$$3^{-n}c<\sum_{j=1}^{d} m(B(x_j,r(x_j)))<\sum_{j=1}^{d} k|\lambda|(B(x_j,r(x_j)))\leq k|\lambda|(V_\varepsilon)\leq k|\lambda|(U_\varepsilon)<k\varepsilon.$$

Hence $c<3^n k\varepsilon$. Since this held for arbitrary $c<m(V_\varepsilon)$, we see that $m(F_k)\leq m(V_\varepsilon)<3^n k\varepsilon$. Letting $\varepsilon\to 0$ shows $m(F_k)=0$. $\square$

3.5 Functions of Bounded Variation

In this section, we will analyze regular signed or complex Borel measures on $\mathbb{R}$ $(n=1)$. We have actually already conducted this analysis for regular positive Borel measures on $\mathbb{R}$ back in Section 1.5. Indeed, Theorem 1.16 implies any such measure μ is of the form μ_F for $F:\mathbb{R}\to\mathbb{R}$ increasing and right-continuous. Conversely, Theorems 1.16 and 1.18 imply any increasing right-continuous function $F:\mathbb{R}\to\mathbb{R}$ yields a regular Borel measure μ_F. One of our objectives is to extend this correspondence to regular complex Borel measures and a new class of $\mathbb{C}$-valued functions. This correspondence will yield decomposition theorems for such functions paralleling those we obtained for measures. Another objective is to understand which $\mathbb{C}$-valued functions F yield complex measures ν_F satisfying $\nu_F \perp m$ and $\nu_F \ll m$. This will provide new insights into Theorem 3.24 in the $n=1$ case, where genuine derivatives now appear:

Ex Given a function $F:\mathbb{R}\to\mathbb{C}$ suppose there exists a regular complex Borel measure ν_F satisfying

$$\nu_F((a,b]) = F(b) - F(a)$$

for $a,b\in\mathbb{R}$ with $a<b$. Let $\nu_F = \lambda + \rho$ be the Lebesgue decomposition with respect to m. Observe that

$$F(x+h)-F(x) = \begin{cases} \nu_F((x,x+h]) & \text{if } h>0 \\ -\nu_F((x+h,x]) & \text{if } h<0 \end{cases}$$

and the families $\{(x,x+h]: h>0\}$ and $\{(x+h,x]: h<0\}$ shrink nicely to x as $|h|\to 0$. Thus Theorem 3.24 says for m-almost every $x\in\mathbb{R}$ that

$$\lim_{h\searrow 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\searrow 0} \frac{\nu_F((x,x+h])}{m((x,x+h])} = \frac{d\rho}{dm}(x)$$

and

$$\lim_{h\nearrow 0} \frac{F(x+h)-F(x)}{h} = \lim_{h\nearrow 0} \frac{\nu_F((x+h,x])}{m((x+h,x])} = \frac{d\rho}{dm}(x)$$

Thus F is differentiable m-almost everywhere with $F' = \frac{d\rho}{dm}$. If $\nu_F \ll m$ so that $\rho = \nu_F$, then

$$\int_{(a,b]} F'\,dm = \int_{(a,b]} \frac{d\nu_F}{dm}\,dm = \nu_F((a,b]) = F(b)-F(a)$$

is a generalization of the fundamental theorem of calculus (or one of its other parts). $\square$

Theorem 3.26 Let $F:\mathbb{R}\to\mathbb{R}$ be increasing, and define $G:\mathbb{R}\to\mathbb{R}$ by

$$G(x) := \lim_{a\searrow x} F(a) = \inf_{a>x} F(a)$$

1. G is increasing and right-continuous.
2. F has a countable discontinuity set.
3. F and G are differentiable m-almost everywhere with $F' = G'$ m-a.e.

Proof 1: For $x<y$, the containment $(x,\infty)\supset(y,\infty)$ implies $G(x)\le G(y)$. Also

$$\inf_{a>x} G(a) = \inf_{a>x}\inf_{b>a} F(b) = \inf_{b>x} F(b) = G(x)$$

implies G is right-continuous.

2: Since F is increasing, for $x\in\mathbb{R}$

$$F(x^-) := \sup_{a<x} F(a) \le F(x) \le \inf_{a>x} F(x) =: F(x^+)$$

and F is discontinuous iff $F(x^-) < F(x^+)$. For $N \in \mathbb{N}$, let $D_N := \{x \in (-N,N) : F \text{ is discontinuous at } x\}$. Then each $x \in D_N$ satisfies $F(-N) \le F(x^-) \le F(x^+) \le F(N)$. Moreover if $x, x' \in D_N$ are distinct, say with $x < x'$, then $F(x^+) \le F(x'^-)$. Thus

$$\bigcup_{x \in D_N} (F(x^-), F(x^+)) \subset [F(-N), F(N)]$$

and the union is disjoint. Hence

$$\sum_{x \in D_N} F(x^+) - F(x^-) \le F(N) - F(-N)$$

Since each term in the sum is positive, D_N is necessarily countable. Therefore $\bigcup_{N \in \mathbb{N}} D_N$ is countable.

3: Let μ_G be the unique Borel measure on $\mathbb{R}$ satisfying $\mu_G((a,b]) = G(b) - G(a)$. Then μ_G is regular by Theorem 1.18 and the example preceding the theorem implies G is differentiable m-almost everywhere. Consider

$$H := G - F$$

Note that $H(x) = 0$ whenever F is continuous at x, and by 2 this fails to hold for countably many points. Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of $\{x \in \mathbb{R} : H(x) \ne 0\}$. Note that

$$F(x^-) \le F(x) \le F(x^+) = G(x)$$

implies $H(x_n) > 0$. Also, arguing as in 2 we have for each $N \in \mathbb{N}$

$$\sum_{|x_n|<N} H(x_n) = \sum_{|x_n|<N} F(x_n^+) - F(x_n) \le \sum_{|x_n|<N} F(x_n^+) - F(x_n^-) \le F(N) - F(-N) < \infty.$$

For each $n \in \mathbb{N}$, let δ_n be the point mass at x_n and define a Borel measure by

$$\mu := \sum_{n \in \mathbb{N}} H(x_n)\, \delta_n.$$

The above estimate shows μ is finite on bounded sets, and hence regular by Theorems 1.16 and 1.18. Moreover, the partition $\mathbb{R} = \{x_n : n \in \mathbb{N}\} \cup \{x_n : n \in \mathbb{N}\}^c$ shows $\mu \perp m$. So by Theorem 3.24 we have for m-almost every $x \in \mathbb{R}$ that

$$\left| \frac{H(x+h) - H(x)}{h} \right| \le \frac{H(x+h) + H(x)}{|h|} \le \frac{\mu((x-2|h|, x+2|h|))}{|h|} = 4 \frac{\mu((x-2|h|, x+2|h|))}{m((x-2|h|, x+2|h|))} \xrightarrow{h \to 0} 0.$$

Thus H is differentiable m-almost everywhere with $H'(x) = 0$. Therefore $F = G - H$ is differentiable m-almost everywhere with $F'(x) = G'(x) - H'(x) = G'(x)$. □

We will now define a class of functions which will correspond to regular complex Borel measures. Recall from Exercise 5 on Homework 8 that for a complex measure ν

$$|\nu|(E) = \sup\left\{ \sum_{j=1}^{n} |\nu(E_j)| : n \in \mathbb{N},\ E = E_1 \cup \cdots \cup E_n \text{ is a partition} \right\}$$

Def For $F : \mathbb{R} \to \mathbb{C}$, the total variation function of F is the function $T_F : \mathbb{R} \to [0,\infty]$ defined by

$$T_F(x) := \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N},\ -\infty < x_0 < x_1 < \cdots x_n = x \right\},$$

and the total variation of F on $\mathbb{R}$ is the quantity $\sup_{x\in\mathbb{R}} T_F(x)$. We say F is of bounded variation on $\mathbb{R}$ its total variation is finite (equivalently, T_F is bounded), and we denote by BV the set of all such functions. □

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The relation between F and T_F is like the relation between ν and $|\nu|$ (which we will soon see even more clearly). Adding more points to the partition

$$(-\infty, x] = (-\infty, x_0] \cup (x_0, x_1] \cup \cdots \cup (x_{n-1}, x_n]$$

can only increase the sum

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})|.$$

So if $x<y$, then we can always assume the partitions of $(-\infty, y]$ include x, and consequently

$$T_F(y) - T_F(x) = \sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n\in\mathbb{N},\ x = x_0 < x_1 < \cdots < x_n = y \right\} \geq 0. \quad *$$

Hence T_F is an increasing function, and the total variation of F equals $\lim_{x\to\infty} T_F(x)$.

$\boxed{\text{Remark}}$ For an increasing function $G:\mathbb{R}\to\mathbb{R}$, we will denote

$$G(-\infty) := \lim_{x\to-\infty} G(x) = \inf_{x\in\mathbb{R}} G(x)$$

$$G(\infty) := \lim_{x\to\infty} G(x) = \sup_{x\in\mathbb{R}} G(x)$$

Thus $F\in BV$ iff $T_F(\infty) < \infty$. □

$\boxed{\text{Ex}}$ 1 Let $F:\mathbb{R}\to\mathbb{R}$ be increasing. Then for $-\infty < x_0 < x_1 < \cdots < x_n = x$ we have

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} F(x_j) - F(x_{j-1}) = F(x) - F(x_0)$$

Thus if $F(-\infty)$ is finite then $T_F(x) = F(x) - F(-\infty)$, and otherwise $T_F \equiv \infty$. In the former case, $T_F(\infty) = F(\infty) - F(-\infty)$ which is finite iff F is bounded. Thus $F\in BV$ iff F is bounded.

2 For $F, G\in BV$ and $a,b\in\mathbb{C}$

$$T_{aF+bG} \leq |a|T_F + |b|T_G$$

Hence $aF+bG\in BV$.

3 Let $a,b\in\mathbb{R}$ with $a\leq b$. Suppose $F:[a,b]\to\mathbb{C}$ satisfies

$$\sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n\in\mathbb{N},\ a = x_0 < x_1 < \cdots < x_n = b \right\} < \infty.$$

Then $\tilde{F}\in BV$ where

$$\tilde{F}(x) := \begin{cases} F(a) & \text{if } x<a \\ F(x) & \text{if } a\leq x\leq b \\ F(b) & \text{if } x>b, \end{cases}$$

since $T_{\tilde{F}}(x)$ equals the above supremum for all $x\geq b$, and so $T_{\tilde{F}}(\infty)$ also equals the above supremum. (Also $T_{\tilde{F}}(x) = 0$ for $x\leq a$). □

Example 3 motivates the following definition:

Def For $a, b \in \mathbb{R}$ with $a<b$ and $F:[a,b] \to \mathbb{C}$, the total variation of F on $[a,b]$ is the quanity

$$\sup\left\{ \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| : n \in \mathbb{N},\ a = x_0 < x_1 < \cdots < x_n = b \right\}.$$

We denote by $BV([a,b])$ the set of functions with finite total variation on $[a,b]$. □

Heuristically, the total variation of F on $[a,b]$ measures the total distance traveled by a particle whose position at time t is $F(t)$.

Ex 4 Suppose $F:[a,b] \to \mathbb{C}$ is differentiable with F' bounded by M, then for any $a = x_0 < x_1 < \cdots < x_n = b$, The mean value theorem implies

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| = \sum_{j=1}^{n} |F'(y_j)|(x_j - x_{j-1}) \le M \sum_{j=1}^{n} (x_j - x_{j-1}) = M(b-a).$$

So $F \in BV([a,b])$. In fact, if F' is Riemann integrable Then the above is a Riemann sum and the total variation equals:

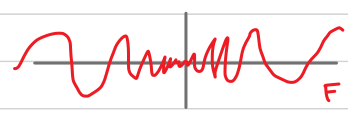
$$\int_a^b |F'(t)|\, dt$$

Recall from calculus that This agrees with the heuristic picture discussed above. In particular, $\sin(x) \in BV([a,b])$ for any finite interval $[a,b]$. However, $\sin(x) \notin BV$ since

$$\sum_{k=1}^{n} \left| \sin\left(\tfrac{(2k+1)\pi}{2}\right) - \sin\left(\tfrac{(2k-1)}{2}\pi\right) \right| = 2n$$

5 Define

$$F(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$



Then $F \notin BV([a,b])$ for any interval containing zero. Exercise check this

6 For $F \in BV$, $F \in BV([a,b])$ for any interval $[a,b] \subset \mathbb{R}$ since (✱) implies the total variation of F on $[a,b]$ is:

$$T_F(b) - T_F(a) \le T_F(b) \le T_F(\infty) < \infty.$$ □

From Examples 3 and 6, we see that everything we will show for functions in BV also applies to functions in $BV([a,b])$ for any interval $[a,b] \subset \mathbb{R}$.

Lemma 3.27 If $F \in BV$ is $\mathbb{R}$-valued, then $T_F + F$ and $T_F - F$ are both increasing.

Proof Let $x<y$ and $\varepsilon>0$. Choose $x_0 < x_1 < \cdots < x_n = x$ so that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \ge T_F(x) - \varepsilon.$$

Then

$$
\begin{aligned}
T_F(y) \pm F(y) &\geq \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \pm F(y) \\
&= \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + |F(y) - F(x)| \pm [F(y) - F(x)] \pm F(x) \\
&\geq T_F(x) - \varepsilon + 0 \pm F(x)
\end{aligned}
$$

Since $\varepsilon > 0$ was arbitrary, we see that $T_F(x) \pm F(x) \leq T_F(y) \pm F(y)$. □

$\boxed{\text{Theorem 3.28}}$ Let $F : \mathbb{R} \to \mathbb{C}$.

1. $F \in BV$ iff $\operatorname{Re} F \in BV$ and $\operatorname{Im} F \in BV$
2. If F is $\mathbb{R}$-valued, then $F \in BV$ iff F is the difference of two bounded increasing functions. In this case, $F = \frac{1}{2}(T_F + F) - \frac{1}{2}(T_F - F)$ and $\frac{1}{2}(T_F \pm F)$ are bounded increasing functions.
3. If $F \in BV$, then $F(x^+) := \lim_{a \searrow x} F(a)$ and $F(x^-) := \lim_{a \nearrow x} F(x)$ exist for all $x \in \mathbb{R}$, as do $F(\pm\infty) := \lim_{a \to \pm\infty} F(a)$.
4. If $F \in BV$, then F has a countable discontinuity set.
5. If $F \in BV$ and $G(x) := F(x^+)$, then $G \in BV$ and F', G' exist and are equal m-almost everywhere.

Proof 1: This follows from

$$T_{\operatorname{Re} F}, T_{\operatorname{Im} F} \leq T_F \leq T_{\operatorname{Re} F} + T_{\operatorname{Im} F}.$$

2: ($\Longrightarrow$) Suppose $F \in BV$, then

$$F = \tfrac{1}{2}(T_F + F) - \tfrac{1}{2}(T_F - F)$$

and by Lemma 3.27 these functions are increasing. We also know T_F is bounded by definition of BV, and so it remains to show F is bounded. Observe that for $x < y$

$$
\begin{aligned}
T_F(x) \pm F(x) \leq T_F(y) \pm F(y) \quad &\Longrightarrow \pm(F(x) - F(y)) \leq T_F(y) - T_F(x) \\
&\Longrightarrow |F(x) - F(y)| \leq T_F(y) - T_F(x) \leq T_F(\infty) < \infty.
\end{aligned}
$$

Hence F is bounded.

($\Longleftarrow$) Suppose $F = G - H$ for $G, H : \mathbb{R} \to \mathbb{R}$ bounded and increasing. Then $G, H \in BV$ by Example 1, and hence $F \in BV$ by Example 2.

3, 4, 5: Using 1 and 2 we can find bounded increasing functions F_1, F_2, F_3, F_4 so that

$$F = F_1 - F_2 + i F_3 - F_4.$$

Then 3 is immediate and 4 follows from Theorem 3.26. Also

$$G(x) = F_1(x^+) - F_2(x^+) + i F_3(x^+) - F_4(x^+).$$

So $G \in BV$ by 1 and 2, and $F' = G'$ m-almost everywhere by Theorem 3.26. □

$\boxed{\text{Def}}$ For an $\mathbb{R}$-valued $F \in BV$, $\frac{1}{2}(T_F + F)$ and $\frac{1}{2}(T_F - F)$ are called the positive and negative variations of F, respectively, and

$$F = \tfrac{1}{2}(T_F + F) - \tfrac{1}{2}(T_F - F)$$

is called the Jordan decomposition of F. □

Observe that $\frac{1}{2}(T_F + F) + \frac{1}{2}(T_F - F) = T_F$, just like for measures.

Remark For $x \in \mathbb{R}$, denote $x^+ := \max\{x, 0\}$ and $x^- := \max\{-x, 0\}$. Then

$$x^+ = \tfrac{1}{2}(|x| + x)$$
$$x^- = \tfrac{1}{2}(|x| - x)$$

and thus for $x_0 < x_1 < \cdots < x_n = x$ and $F : \mathbb{R} \to \mathbb{R}$

$$\sum_{j=1}^{n} [F(x_j) - F(x_{j-1})]^{\pm} = \frac{1}{2} \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \pm (F(x_j) - F(x_{j-1}))$$
$$= \frac{1}{2} \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \pm \frac{1}{2}(F(x) - F(x_0))$$

It follows that

$$\frac{1}{2}(T_F \pm F)(x) = \sup\left\{ \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})]^{\pm} \pm \frac{1}{2} F(x_0) : -\infty < x_0 < x_1 < \cdots < x_n = x \right\}$$
$$= \sup\left\{ \sum_{j=1}^{n} [F(x_j) - F(x_{j-1})]^{\pm} : -\infty < x_0 < x_1 < \cdots < x_n = x \right\} \pm \frac{1}{2} F(-\infty).$$

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Theorem 3.28 tells us we can guarantee right-continuity for $F \in BV$ by replacing $F(x)$ with $F(x^+)$ if necessary, which only modifies F at countably many points. Moreover, the theorem also tells us

$$F = F_1 - F_2 + iF_3 - iF_4$$

for $F_1, \ldots, F_4$ bounded increasing functions. We will see below that the F_j can be chosen to be right-continuous (see Lemma 3.29), and hence there exist unique Borel measures μ_j satisfying $\mu_j((a,b]) = F_j(b) - F_j(a)$ by Theorem 1.16. As F_j is bounded, μ_j is finite and therefore

$$\nu := \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

is a complex Borel measure satisfying $\nu((a,b]) = F(b) - F(a)$. Recall that up to adding a constant, F_j corresponded uniquely to μ_j. So by choosing an appropriate normalization, the correspondence $F \leftrightarrow \nu$ will be one-to-one. We therefore make the following definition:

Def We say $F \in BV$ is normalized if it is right-continuous and $F(-\infty) = 0$, and we denote the set of all such functions by NBV.

Observe that for $F \in BV$, $G(x) := F(x^+) - F(-\infty) \in NBV$ by Theorem 3.28 (and Example 2).

Lemma 3.29 For $F \in BV$, $T_F(-\infty) = 0$. If F is right-continuous then so is T_F. In particular, for $F \in NBV$, $\frac{1}{2}(T_{\operatorname{Re} F} \pm \operatorname{Re} F)$, $\frac{1}{2}(T_{\operatorname{Im} F} \pm \operatorname{Im} F) \in NBV$.

Proof For $x \in \mathbb{R}$ and $\varepsilon > 0$, let $x_0 < x_1 < \cdots < x_n = x$ be such that

$$T_F(x) \le \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| + \varepsilon$$

Since $T_F(x) - T_F(x_0)$ gives the total variation of F on $[x_0, x]$ (see Example 6), it follows that

$$T_F(x) \le T_F(x) - T_F(x_0) + \varepsilon,$$

or $T_F(x_0) \le \varepsilon$. Since T_F is increasing, we have $T_F(y) \le \varepsilon$ for all $y \le x_0$. Thus $T_F(-\infty) = 0$.
Now, suppose F is right-continuous. For $x \in \mathbb{R}$, let $\alpha := T_F(x^+) - T_F(x)$ (which we must show is zero). For $\varepsilon > 0$, let $\delta > 0$ be such that $0 < h < \delta$ implies

$$|F(x+h) - F(x)| < \varepsilon$$

and

$$T_F(x+h) - T_F(x^+) < \varepsilon.$$

Fix $0 < h < \delta$ and let $x = x_0 < x_1 < \cdots < x_n = x+\varepsilon$ be such that

$$\sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \ge \frac{3}{4}\left(T_F(x+h) - T_F(x)\right) \ge \frac{3}{4}\alpha$$

Note that $|F(x_1) - F(x)| < \varepsilon$ since $x < x_1 < x+h$, and so we have

$$\begin{aligned} \frac{3}{4}\alpha &\le \varepsilon + \sum_{j=2}^{n} |F(x_j) - F(x_{j-1})| \\ &\le \varepsilon + T_F(x+h) - T_F(x_1) \\ &\le \varepsilon + T_F(x+h) - T_F(x^+) < 2\varepsilon. \end{aligned}$$

Thus $\alpha < \frac{8}{3}\varepsilon$, and consequently $\alpha = 0$.
Finally, $F \in NBV$ implies $\operatorname{Re} F, \operatorname{Im} F \in NBV$ and the above implies $T_F \in NBV$. Hence the positive and negative variations of $\operatorname{Re} F$ and $\operatorname{Im} F$ belong to NBV. $\square$

Theorem 3.30 For a regular complex Borel measure μ on $\mathbb{R}$, $F_\mu(x) := \mu((-\infty, x]) \in NBV$. Conversely, for any $F \in NBV$ there is a unique regular complex Borel measure μ_F on $\mathbb{R}$ such that $F(x) = \mu((-\infty, x])$. Moreover, one has

$$|\mu_F| = \mu_{T_F}$$

$$(\mu_F)_r = \mu_{\operatorname{Re} F} \qquad\qquad (\mu_F)_i = \mu_{\operatorname{Im} F}$$

$$(\mu_F)_r^{\pm} = \mu_{\frac{1}{2}(T_{\operatorname{Re} F} \pm \operatorname{Re} F)} \qquad\qquad (\mu_F)_i^{\pm} = \mu_{\frac{1}{2}(T_{\operatorname{Im} F} \pm \operatorname{Im} F)}$$

Proof Let $F_r^{\pm}(x) := \mu_r^{\pm}((-\infty, x])$ and $F_i^{\pm}(x) := \mu_i^{\pm}((-\infty, x])$. Then $F_r^{\pm}, F_i^{\pm}$ are increasing and bounded (since $\mu_r^{\pm}, \mu_i^{\pm}$ are finite), and hence belong to BV by Example 1. Thus

$$F_\mu(x) := \mu((-\infty, x]) = (\mu_r^+ - \mu_r^- + i\mu_i^+ - i\mu_i^-)((-\infty, x]) = F_r^+(x) - F_r^-(x) + iF_i^+(x) - iF_i^-(x) \in BV$$

by Theorem 3.28. Moreover, $F_r^{\pm}$ and $F_i^{\pm}$ are right-continuous and zero at $-\infty$ by continuity from above. Hence $F_\mu \in NBV$.
Conversely, for $F \in NBV$ we have

$$F = \underbrace{\tfrac{1}{2}(T_{\operatorname{Re} F} + \operatorname{Re} F)}_{=:F_1} - \underbrace{\tfrac{1}{2}(T_{\operatorname{Re} F} - \operatorname{Re} F)}_{=:F_2} + i\underbrace{\tfrac{1}{2}(T_{\operatorname{Im} F} + \operatorname{Im} F)}_{=:F_3} - i\underbrace{\tfrac{1}{2}(T_{\operatorname{Im} F} - \operatorname{Im} F)}_{=:F_4}$$

and $F_1, F_2, F_3, F_4 \in NBV$ by Lemma 3.29. For each $j = 1, 2, 3, 4$, let μ_j be the unique Borel measure from Theorem 1.16 satisfying $\mu_j((a,b]) = F_j(b) - F_j(a)$. Since $F_j(-\infty) = 0$, it follows that $\mu_j((-\infty, x]) = F_j(x)$. Also note that each μ_j is regular by Theorem 1.18, and finite since F_j is bounded. Hence

$$\mu_F := \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

defines a regular complex Borel measure satisfying $\mu_F((-\infty, x]) = F(x)$. The uniqueness follows from the uniqueness of $\mu_1, \mu_2, \mu_3, \mu_4$.

Finally, we must show $|\mu_F| = \mu_{T_F}$. (The remaining formulas then follow directly from this.) First observe that by Exercise 5 on Homework 8

$$\mu_{T_F}((-\infty,x]) = T_F(x) = \sup\Big\{\sum_{j=1}^{n} \underbrace{|F(x_j) - F(x_{j-1})|}_{\mu_F((x_{j-1},x_j])} : n \in \mathbb{N}, x_0 < x_1 < \cdots < x_n = x\Big\}$$

$$\le \sup\Big\{\sum_{j=1}^{n} |\mu_F(E_j)| : n \in \mathbb{N}, (-\infty,x] = E_1 \cup \cdots \cup E_n \text{ is a partition}\Big\} = |\mu_F|((-\infty,x]).$$

On the other hand, for disjoint intervals $(a_n, b_n) \subset \mathbb{R}$, $n \in \mathbb{N}$, we have by Exercise 5 on Homework 3

$$\Big|\mu_F\Big(\bigcup_{n=1}^{\infty}(a_n,b_n)\Big)\Big| \le \sum_{n=1}^{\infty} |F(b_n^-) - F(a_n)| \le \sum_{n=1}^{\infty} T_F(b_n^-) - T_F(a_n) = \mu_{T_F}\Big(\bigcup_{n=1}^{\infty}(a_n,b_n)\Big)$$

The regularity of μ_F and μ_{T_F} therefore implies $|\mu_F(E)| \le \mu_{T_F}(E)$ for all Borel sets $E \subset \mathbb{R}$. Thus if $(-\infty,x] = E_1 \cup \cdots \cup E_n$ is a partition of by Borel sets, then

$$\sum_{j=1}^{n} |\mu_F(E_j)| \le \sum_{j=1}^{n} \mu_{T_F}(E_j) = \mu_{T_F}((-\infty,x]).$$

So $|\mu_F|((-\infty,x]) \le \mu_{T_F}((-\infty,x])$ by Exercise 5 on Homework 8 again. Hence $|\mu_F|$ and μ_{T_F} agree on $(-\infty,x]$, and so are equal by the uniqueness of μ_{T_F}. $\square$

This completes the first objective of this section. We now move on to understanding which functions $F \in NBV$ satisfy $\mu_F \perp m$ or $\mu_F \ll m$.

Proposition 3.31 If $F \in NBV$ then $F' \in L^1(\mathbb{R}, m)$. Moreover, $\mu_F \perp m$ if and only if $F' = 0$ m-almost everywhere, and $\mu_F \ll m$ if and only if $F(x) = \int_{(-\infty,x]} F'\,dm$ for all $x \in \mathbb{R}$.

Proof Let $\mu_F = \lambda + \rho$ be the Lebesgue decomposition with respect to m, where $\lambda \perp m$ and $\rho \ll m$. By Example I, $F' = \frac{d\rho}{dm} \in L^1(\mathbb{R}, m)$ since ρ is finite. We have $\mu_F \perp m$ iff $\mu_F = \lambda$ iff $\rho \equiv 0$ iff $F' = \frac{d\rho}{dm} = 0$ m-a.e. Also, $\mu_F \ll m$ iff $\mu_F = \rho$ iff $d\mu_F = F'\,dm$. The last statement is equivalent to

$$F(x) = \mu_F((-\infty,x]) = \int_{(-\infty,x]} F'\,dm$$

by the uniqueness of μ_F. $\square$

There is another way to characterize $\mu_F \ll m$ that does not rely on computing F' or its integrals. To motivate it, recall that $\mu_F \ll m$ iff $|\mu_F| \ll m$, and by Theorem 3.5 this is further equivalent to: $\forall \varepsilon > 0$ $\exists \delta > 0$ so that $m(E) < \delta$ implies $|\mu_F|(E) < \varepsilon$. In particular, if $E = (a_1,b_1) \cup \cdots \cup (a_n,b_n)$ is a disjoint union, then

$$\sum_{j=1}^{n} (b_j - a_j) = m(E) < \delta$$

implies by Exercise 5 on Homework 8 that

$$\sum_{j=1}^{n} |F(b_j) - F(a_j)| = \sum_{j=1}^{n} |\mu_F((a_j,b_j])| \le |\mu_F|(E) < \varepsilon.$$

Note that we have used $\mu_F \ll m$ to deduce $\mu_F((a,b)) = \mu_F((a,b]) = F(b) - F(a)$.

Def We say $F:\mathbb{R}\to\mathbb{C}$ is absolutely continuous if for every $\varepsilon>0$ there exists $\delta>0$ so that for any disjoint intervals $(a_1,b_1),\dots,(a_n,b_n)\subset\mathbb{R}$

$$\sum_{j=1}^{n}(b_j-a_j)<\delta \implies \sum_{j=1}^{n}|F(b_j)-F(a_j)|<\varepsilon.$$

For $F:[a,b]\to\mathbb{C}$, we say F is absolutely continuous on $[a,b]$ if the above holds for $(a_j,b_j)\subset[a,b]$. □

Note that absolute continuity implies uniform continuity by considering $n=1$ in the above definition. We have shown above that any $F\in NBV$ with $\mu_F\ll m$ is absolutely continuous, and the converse will be shown in Proposition 3.32 below.

Ex Suppose $F:\mathbb{R}\to\mathbb{C}$ is differentiable with bounded derivative. Then the mean value theorem implies

$$\sum_{j=1}^{n}|F(b_j)-F(a_j)| \le \sum_{j=1}^{n}\sup_{x\in\mathbb{R}}|F'(x)|\,|b_j-a_j|.$$

So given $\varepsilon>0$, choosing $\delta:=\frac{\varepsilon}{\sup|F'|}$ shows F is absolutely continuous. □

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Proposition 3.32 For $F\in NBV$, $\mu_F\ll m$ if and only if F is absolutely continuous.

Proof We have already shown the "only if" direction, so suppose F is absolutely continuous. Let $E\subset\mathbb{R}$ be a Borel set with $m(E)=0$, let $\varepsilon>0$, and let $\delta>0$ be as in the definition of absolute continuity. Using the regularity of m and μ_F, we can find $U\supset E$ open such that $m(U)<\delta$ and $|\mu_F(U)-\mu_F(E)|\le\varepsilon$. Since it is open, U is a countable union of disjoint open intervals:

$$U=\bigcup_{j=1}^{\infty}(a_j,b_j).$$

Then for $n\in\mathbb{N}$, $\sum_{j=1}^{n}(b_j-a_j)\le m(U)<\delta$ implies $\sum_{j=1}^{n}|F(b_j)-F(a_j)|<\varepsilon$. Letting $n\to\infty$ yields:

$$|\mu_F(U)|\le\sum_{j=1}^{\infty}|F(b_j)-F(a_j)|\le\varepsilon.$$

Hence $|\mu_F(E)|\le|\mu_F(E)-\mu_F(U)|+|\mu_F(U)|\le 2\varepsilon$. Letting $\varepsilon\to 0$ shows $\mu_F(E)=0$ and so $\mu_F\ll m$. □

Combining the last part of Proposition 3.31 with Proposition 3.32 gives

Corollary 3.33 For $f\in L^1(\mathbb{R},m)$, $F(x):=\int_{(-\infty,x]}f\,dm\in NBV$ and is absolutely continuous. Conversely, if $F\in NBV$ is absolutely continuous, then $F'\in L^1(\mathbb{R},m)$ with $F(x)=\int_{(-\infty,x]}F'\,dm$.

If we restrict ourselves to an interval $[a,b]\subset\mathbb{R}$, then the hypothesis that $F\in NBV$ can be dropped in the above corollary. We first need a lemma:

Lemma 3.34 If F is absolutely continuous on $[a,b]$, then $F\in BV([a,b])$.

Proof For $\varepsilon=1$, let $\delta>0$ be as in the definition of absolute continuity. Fix $N\in\mathbb{N}$ with $N\ge\frac{1}{\delta}$, and let $t_k:=a+\frac{k}{N}(b-a)$ for $k=0,1,\dots,N$. Given $a=x_0<x_1<\cdots<x_N=b$, the sum $\sum|F(x_j)-F(x_{j-1})|$ is only made larger by adding $t_0,t_1,\dots,t_N$, and in this case we have

$$\sum_{j=1}^{n} |F(x_j)-F(x_{j-1})| = \sum_{k=1}^{N} \sum_{t_{k-1}=x_j<t_k} |F(x_j)-F(x_{j-1})| \le \sum_{k=1}^{N} 1 = N.$$

Thus the total variation of F on $[a,b]$ is bounded by N. □

Theorem 3.35 (The Fundamental Theorem for Lebesgue Integrals)
For $[a,b]\subset\mathbb{R}$ and $F:[a,b]\to\mathbb{C}$, the following are equivalent:

1. F is absolutely continuous on $[a,b]$
2. There exists $f\in L^1([a,b],m)$ so that
$$F(x) = F(a) + \int_{[a,x]} f\,dm$$
3. F is differentiable m-almost everywhere on $[a,b]$ with $F'\in L^1([a,b],m)$, and $F(x)=F(a)+\int_{[a,x]}F'\,dm$.

Proof 3 $\Rightarrow$ 2 is immediate, and 2 $\Rightarrow$ 1 follows from Corollary 3.33 after extending f to $\mathbb{R}$ by letting $f(t)=0$ for all $t\notin[a,b]$.
1 $\Rightarrow$ 3: By Lemma 3.34, $F\in BV([a,b])$. And so extending F to $\mathbb{R}$ as in Example 3, we have $F\in BV$. Since F is continuous by virtue of being absolutely continuous, it follows that $F-F(a)\in NBV$. Hence Corollary 3.33 implies $F'=(F-F(a))'\in L^1(\mathbb{R},m)$ with
$$F(x)-F(a) = \int_{(\infty,x]} (F-F(a))'\,dm = \int_{(\infty,x]} F'\,dm.$$
□

Def Let $F\in NBV$, and let μ_F be the unique regular complex Borel measure satisfying $F(x)=\mu_F((-\infty,x])$. For $g\in L^1(\mathbb{R},\mu_F)$, we denote
$$\int_{\mathbb{R}} g\,dF := \int_{\mathbb{R}} g\,d\mu_F,$$
and call this a Lebesgue-Stieltjes integral of g with respect to F. □

The final results of this section show that the Lebesgue Stieltjes integrals satisfy an integration-by-parts formula:

Theorem 3.36 Let $F,G\in NBV$ with at least one of them continuous. For $a,b\in\mathbb{R}$ with $a<b$,
$$\int_{(a,b]} F\,dG + \int_{(a,b]} G\,dF = F(b)G(b) - F(a)G(a)$$

Proof By decomposing F and G into linear combinations of bounded increasing functions (and μ_F and μ_G into positive measures), we may assume F and G are $\mathbb{R}$-valued, bounded, and increasing. Now, without loss of generality, suppose G is continuous. Consider the set
$$\Omega := \{(x,y)\in\mathbb{R}^2 : a<x\le y<b\}.$$
Using Tonelli's theorem (Theorem 2.36), we have
$$\mu_F\times\mu_G(\Omega) = \int_{(a,b]}\int_{(a,b]} I_\Omega(x,y)\,dF(x)\,dG(y) = \int_{(a,b]}\int_{(a,y]} dF\,dG(y)$$
$$= \int_{(a,b]} F(y)-F(a)\,dG(y) = \int_{(a,b]} F\,dG - F(a)[G(b)-G(a)],$$
and computing the iterated integral in the other order gives:

$$\mu_F \times \mu_G(\Omega) = \int_{[a,b]}\int_{[a,b]} I_\Omega(x,y)\, dG(y)\, dF(x) = \int_{[a,b]}\int_{[x,b]} dG\, dF(x)$$
$$= \int_{[a,b]} G(b) - G(x)\, dF(x) = G(b)[F(b)-F(a)] - \int_{[a,b]} G\, dF,$$
where we have used the continuity of G to assert $G(x^-)=G(x)$. Subtracting these two computations and rearranging terms yields the claimed equality. □

Cor 3.37 If F and G are absolutely continuous on $[a,b]$, then
$$\int_{[a,b]} FG' + F'G\, dm = F(b)G(b) - F(a)G(a)$$

Proof First observe that
$$\int_{[a,b]} (F-F(a))(G-G(a))' + (F-F(a))'(G-G(a))\, dm = \int_{[a,b]} FG' - F(a)G' + F'G - F'G(a)\, dm$$
$$= \int_{[a,b]} FG' + F'G\, dm - F(a)(G(b)-G(a)) - (F(b)-F(a))G(a)$$
and
$$(F(b)-F(a))(G(b)-G(a)) = F(b)G(b) - F(b)G(a) - F(a)(G(b)-G(a))$$
$$= F(b)G(b) - F(a)G(a) - (F(b)-F(a))G(a) - F(a)(G(b)-G(a))$$
So it suffices to assume $F(a)=G(a)=0$. Lemma 3.34 implies $F, G \in BV([a,b])$ and extending them to $\mathbb{R}$ as in Example 3 we have $F, G \in NBV$. Since both are continuous, Theorem 3.36 implies
$$F(b)G(b) = \int_{[a,b]} F\, dG + \int_{[a,b]} G\, dF = \int_{[a,b]} FG' + GF'\, dm,$$
where we have used $F' = \frac{d\mu_F}{dm}$ and $G' = \frac{d\mu_G}{dm}$. □