Midterm 2

Name: ____

NetID: ____

Instructions:

1. You will have 50 minutes to complete the exam.
2. The exam is a total of 3 questions, their respective points values are listed below.
3. Unless stated otherwise, you must justify your answers with proofs.
4. You may cite any results from lecture or the homework.
5. No books, notes, calculators, or electronic devices are permitted.
6. If you require additional space, please use the reverse side of the pages.
7. The exam has a total of 4 pages, please verify that your copy has all 4 pages.

Question	**Score**	**Points**
1.		20
2.		10
3.		20
Total		50

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1. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{M}$-measurable functions. Suppose $(f_n)_{n\in\mathbb{N}}$ converges in measure to an $\mathcal{M}$-measurable function f, and suppose there exists $R > 0$ so that $|f_n(x)| \leq R$ for μ-almost every $x \in X$ and every $n \in \mathbb{N}$.

   (a) Show that $|f(x)| \leq R$ for μ-almost every $x \in X$.

   (b) Show that $f_n, f \in L^1(X, \mu)$ for all $n \in \mathbb{N}$ and that $f_n \to f$ in L^1.

2. Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $f\colon X \to [0,\infty)$ be an $\mathcal{M}$-measurable function. Consider the following subset of the product space $X \times [0,\infty)$:

$$E := \{(x, f(x))\colon x \in X\}.$$

Show that $\mu \times m(E) = 0$, where m is the Lebesgue measure on $[0,\infty)$.

[**Hint:** use a fine partition of the range to obtain a countable cover of E by measurable rectangles.]

3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, and suppose $(\nu_n)_{n\in\mathbb{N}}$ is a sequence of positive measures satisfying $\nu_n(E) \leq \nu_{n+1}(E) \leq \mu(E)$ for all $E \in \mathcal{M}$.

   (a) For each $n \in \mathbb{N}$ show that $\nu_n \ll \mu$ and $0 \leq \frac{d\nu_n}{d\mu} \leq \frac{d\nu_{n+1}}{d\mu} \leq 1$ μ-almost everywhere.

   (b) If $f := \sup_n \frac{d\nu_n}{d\mu}$ and $d\nu := f d\mu$, show that ν is finite and $\nu_n \to \nu$ in total variation norm.