III.1 Elementary Properties and Examples

Def An algebra over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ that admits a multiplication operation

$$A \times A \to A$$
$$(a,b) \mapsto a \cdot b$$

satisfying

1. $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ for all $a,b,c \in A$ (associativity)
2. $a\cdot(b+c) = a\cdot b + a\cdot c$
   $(a+b)\cdot c = a\cdot c + b\cdot c$ for all $a,b,c\in A$ (distributivity)
3. $\alpha(ab) = (\alpha a)b = a(\alpha b)$ for all $\alpha\in\mathbb{F}$ and $a,b\in A$.

We say A is unital if it admits an element $e\in A$ satisfying $a\cdot e = e\cdot a = a$ $\forall a\in A$.
We say A is abelian if $a\cdot b = b\cdot a$ for all $a,b\in A$.

Def A Banach algebra is an algebra A over a field $\mathbb{F}$ equipped with a norm making A into a Banach space and such that

$$\|a\cdot b\| \le \|a\|\,\|b\| \qquad \forall a,b\in A.$$

If A is unital, then it is assumed that $\|e\| = 1$.

- For a unital Banach algebra, $\|\alpha e\| = |\alpha|$ implies $\mathbb{F}\ni\alpha\mapsto \alpha e$ is an isometry. In this case, we identify $\mathbb{F}\subseteq A$ and denote the unit by 1.

Ex
1. For X a Hausdorff space, $C_b(X)$ is a unital abelian Banach algebra with unit given by the constant function 1.
2. For X a locally compact Hausdorff space, $C_0(X)$ is an abelian Banach algebra. It is unital iff X is compact.
   In particular, $c_0(\mathbb{N})$ is a Banach algebra.
3. For (X,Ω,μ) a σ-finite measure space, $L^\infty(X,\Omega,\mu)$ is a unital abelian Banach algebra with unit $1(x) = 1$ μ-a.e.
4. For $\mathcal{H}$ a Hilbert space, $B(\mathcal{H})$ is a unital Banach algebra when equipped with the operator norm. It is abelian iff $\dim\mathcal{H} = 1$.
   Also $K(\mathcal{H})$, the compact operators are a Banach algebra.
   In particular, $M_{n\times n}(\mathbb{C})$ is a unital Banach algebra for all $n\in\mathbb{N}$. $\square$

- If A is a non-unital Banach algebra, let $A_1 := A\times\mathbb{F}$ and define
  1. $(a,\alpha) + (b,\beta) = (a+b, \alpha+\beta)$
  2. $\beta(a,\alpha) = (\beta a, \beta\alpha)$

③ $(a,\alpha)\cdot(b,\beta) = (ab+\alpha b+\beta a, \alpha\beta)$

④ $\|(a,\alpha)\| = \|a\| + |\alpha|$

Then A_1 is a unital Banach algebra with unit $(0,1)$. Moreover $a \mapsto (a,0)$ is an isometric embedding of A into A_1 (Exercise)

Def A_1 defined above is called the unitization of A.

Ex Let X be a locally compact Hausdorff space that is not compact. Let X_∞ be its one-point compactification. Then the unitization of $C_0(X)$ is $C(X_\infty)$.

Ex ① For $f,g \in L^1(\mathbb{R})$ define their convolution by

$$(f*g)(t) := \int_{\mathbb{R}} f(t-s)g(s)\,ds$$

Observe that

$$\begin{aligned}\int_{\mathbb{R}} |f*g(t)|\,dt &\le \iint_{\mathbb{R}^2} |f(t-s)|\,|g(s)|\,ds\,dt \\ &= \int_{\mathbb{R}} |g(s)|\left(\int_{\mathbb{R}} |f(t-s)|\,dt\right)ds \\ &= \int_{\mathbb{R}} g(s)\,\|f\|_1\,ds = \|f\|_1\|g\|_1.\end{aligned}$$

Thus $f*g \in L^1(\mathbb{R})$ with $\|f*g\|_1 \le \|f\|_1\|g\|_1$. Thus $L^1(\mathbb{R})$ is a Banach algebra when equipped with convolution. It is abelian but non-unital (Exercise)

② For $\mu,\nu \in M(\mathbb{R})$, define $\mu*\nu \in M(\mathbb{R}) = C_0(\mathbb{R})^*$ by

$$(\mu*\nu)(f) = \iint f(t+s)\,d\mu(t)\,d\nu(s)$$

Observe

$$|(\mu*\nu)(f)| \le \iint |f(t+s)|\cdot d|\mu|(t)\,d|\nu|(s) \le \|f\|_\infty\|\mu\|\cdot\|\nu\|.$$

So $\|\mu*\nu\| \le \|\mu\|\cdot\|\nu\|$. It follows that $M(\mathbb{R})$ is a Banach algebra with this convolution operation. It is abelian and unital with unit δ_0 (Exercise)

Moreover, $L^1(\mathbb{R})$ is a Banach subalgebra of $M(\mathbb{R})$: for $g \in L^1(\mathbb{R})$ define $\mu_g \in M(\mathbb{R})$ by

$$\mu_g(f) := \int f(t)g(t)\,dt \qquad f \in C_0(\mathbb{R})$$

Then

$$\begin{aligned}(\mu_g*\mu_h)(f) &= \iint f(t+s)g(t)h(s)\,dt\,ds = \iint f(t)\,g(t-s)\,h(s)\,dt\,ds \\ &= \int f(t)\,(g*h)(t-s)\,dt = \mu_{g*h}(f).\end{aligned}$$

So $\mu_g*\mu_h = \mu_{g*h}$. $\square$

VII.3 The Spectrum

Def An elemental $a \in A$ of a unital algebra is said to be *invertible* if there exists $b \in A$ satisfying $ab = ba = 1$. In this case, we denote $a^{-1} := a$.

Ex ① Let X be a compact Hausdorff space. Then $f \in C(X)$ is invertible iff $f(x) \neq 0$ $\forall x \in X$, in which case $(f^{-1})(x) = 1/f(x)$ with $\|f^{-1}\| = (\inf_{x \in X} |f(x)|)^{-1}$.

⚠ not compositional inverse.

② Let (X, Ω, μ) be a σ-finite measure space. Then $f \in L^\infty(X,\mu)$ is invertible iff

$$\delta := \inf\{\varepsilon > 0 : \mu(\{x \in X : |f(x)| \leq \varepsilon\}) = 0\} > \delta$$

In this case, $(f^{-1})(x) = 1/f(x)$ and $\|f\|_\infty = \delta$.

③ Let $\mathcal{H}$ be a Hilbert space. Then $T \in B(\mathcal{H})$ is invertible as a Banach space element iff it is bijective (using the Inverse Mapping Theorem). □

Lemma If A is a unital Banach algebra and $x \in A$ satisfies $\|x - 1\| < 1$, then x is invertible with $x^{-1} = \sum_{n=0}^{\infty} (1-x)^n$

Proof Define $y := 1 - x$ and $r := \|y\|$. It follows from an induction argument that $\|y^n\| \leq \|y\|^n = r^n$. Consequently

$$\sum_{n=0}^{\infty} \|y^n\| \leq \frac{1}{1-r} < \infty,$$

which implies $z := \sum_{n=0}^{\infty} y^n$ converges to an element of A. Denote $z_N := \sum_{n=0}^{N} y^n$ so that $\|z - z_N\| \to 0$. Observe that

$$z_N x = (1 + y + \cdots + y^N)(1-y) = (1 + y + \cdots + y^N) - (y + y^2 + \cdots + y^{N+1}) = 1 - y^{N+1}.$$

Also note that $\|y^{N+1}\| \leq r^{N+1}$ implies $y^{N+1} \to 0$. Thus

$$zx = \lim_{N\to\infty} z_N x = \lim_{N \to \infty} (1 - y^{N+1}) = 1.$$

Similarly, $xz = 1$. Thus $z = x^{-1}$. □

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Thm Let A be a unital Banach algebra. Then

$$G := \{a \in A : a \text{ is invertible}\}$$

is open, the map

$$G \ni a \mapsto a^{-1} \in G$$

is continuous.

Proof Let $a \in G$. Suppose $b \in A$ satisfies $\|a - b\| < \frac{1}{\|a^{-1}\|}$. Then we have

$$1 = \frac{1}{\|a^{-1}\|} \cdot \|a^{-1}\| > \|a - b\| \cdot \|a^{-1}\| \geq \begin{cases} \|1 - ba^{-1}\| \\ \|1 - a^{-1}b\| \end{cases}$$

So the lemma implies ba^{-1} and $a^{-1}b$ are invertible. We claim that $b \in G$ with $b^{-1} = a^{-1}(ba^{-1})^{-1} = (a^{-1}b)^{-1}a^{-1}$. Indeed

$$(*) \quad b\left(a^{-1}(ba^{-1})^{-1}\right) = (ba^{-1})\cdot(ba^{-1})^{-1} = I$$
$$\left((a^{-1}b)^{-1}a^{-1}\right)\cdot b = (a^{-1}b)^{-1}\cdot(a^{-1}b) = I$$

Consequently

$$(a^{-1}b)^{-1}a^{-1} = (a^{-1}b)^{-1}a^{-1}\cdot I = (a^{-1}b)^{-1}a^{-1}\cdot b\cdot\left(a^{-1}(ba^{-1})^{-1}\right) = 1\cdot a^{-1}(ba^{-1})^{-1} = a^{-1}(ba^{-1})^{-1}.$$

This equality along with $(*)$ completes the claim. Thus

$$B(a, 1/\|a^{-1}\|) \subset G,$$

and G is open.

Toward checking continuity of $a \mapsto a^{-1}$, we will first check continuity at $1 \in G$. Let $\varepsilon > 0$. Choose $\delta \in (0,1)$ satisfying $\frac{\delta}{1-\delta} < \varepsilon$. If $\|a - 1\| < \delta < 1$, then the lemma implies $a^{-1} = \sum_{n=0}^{\infty} (1-a)^n$. Consequently

$$\|a^{-1} - 1\| = \left\|\sum_{n=1}^{\infty}(1-a)^n\right\| \leq \sum_{n=1}^{\infty}\|1-a\|^n < \sum_{n=1}^{\infty}\delta^n = \frac{\delta}{1-\delta} < \varepsilon.$$

Thus $a \mapsto a^{-1}$ is continuous at 1.

Now, suppose $(a_n)_{n\in\mathbb{N}} \subset G$ converges to some $a_0 \in G$. Then $a_0^{-1}a_n \to 1$ and so the above implies $a_n^{-1}a_0 = (a_0^{-1}a_n)^{-1} \to 1$. Consequently, $a_n^{-1} \to a_0^{-1}$. □

Def Let A be a unital Banach algebra. The <u>spectrum</u> of $a \in A$ is the set

$$\sigma(a) := \{\alpha \in \mathbb{F} : a - \alpha \text{ is not invertible}\}$$

The <u>resolvent</u> of a is the set $\rho(a) := \mathbb{F} \setminus \sigma(a)$.

Ex ① Let X be a compact Hausdorff space. For $f \in C(X)$, $\sigma(f) = f(X)$. Indeed, $f - \alpha$ is invertible iff $(f-\alpha)(x) = f(x) - \alpha \neq 0 \ \forall x \in X \iff \alpha \notin f(X)$.

② Let $\mathcal{H}$ be a Hilbert space. For $T \in B(\mathcal{H})$, $\sigma_p(T) \subseteq \sigma(T)$. Indeed, $\lambda \in \sigma_p(T)$ implies $\ker(T - \lambda) \neq \{0\}$, and so $T - \lambda$ is not invertible. Generally, though, $\sigma_p(T) \subsetneq \sigma(T)$. E.g. $S \in B(\ell^2(\mathbb{N}))$ the shift operator has $0 \in \sigma(S) \setminus \sigma_p(S)$ since $S - 0 = S$ is injective but not surjective.

<u>Exercise</u>: Let $T \in K(\mathcal{H})$ be a normal operator. Show that

$$\sigma_p(T) \subseteq \sigma(T) \subseteq \sigma_p(T) \cup \{0\}.$$

Moreover, if $|\sigma_p(T)| < \infty$, then $\sigma_p(T) = \sigma(T)$.

③ Consider $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in M_2(\mathbb{R})$. Then the previous example implies $\sigma(A) = \sigma_p(A) = \emptyset$ since A has no real eigenvalues. However, viewing $A \in M_2(\mathbb{C})$ we obtain $\sigma(A) = \{\pm i\}$. $\square$

- As indicated by the above example, taking our Banach algebras over $\mathbb{C}$ yields non-empty spectra. To see this, we first need a definition:

Def Let X be a Banach space. An X-valued analytic function is a map $f: G \to X$ where $G \subseteq \mathbb{C}$ is open and for all $z_0 \in G$ the limit

$$f'(z_0) := \lim_{h \to 0} \frac{1}{h}[f(z_0+h) - f(z_0)]$$

exists, and $f': G \to X$ is continuous.

- Exercise ⓐ Show that $f: G \to X$ is analytic iff $x^* \circ f: G \to \mathbb{C}$ is analytic $\forall x^* \in X^*$.

ⓑ Show that $f: G \to X$ is analytic iff $\forall z_0 \in G$ $\exists x_0, x_1, x_2, \ldots \in X$ such that

$$f(z) = \sum_{n=0}^{\infty} x_n (z - z_0)^n \qquad \forall z \in B(z_0, \operatorname{dist}(z_0, \partial G))$$

- Many results in complex analysis still hold for X-valued analytic functions, including the Cauchy integral formula and (consequently) Liouville's Theorem.

Thm Let A be a unital Banach algebra over $\mathbb{C}$. For all $a \in A$, $\sigma(a)$ is a non-empty compact subset of $\{z \in \mathbb{C} : |z| \le \|a\|\}$. Moreover, $\rho(a) \ni z \mapsto (z-a)^{-1}$ is an A-valued analytic function.

Proof Suppose $|z| > \|a\|$. Then $1 - \frac{1}{z}a$ is invertible by the earlier lemma. Since $z \neq 0$, it follows that $z(1 - \frac{1}{z}a) = z - a$ is invertible. Hence $z \in \rho(a)$. This shows $\sigma(a) \subseteq \{z \in \mathbb{C} : |z| \le \|a\|\}$. Consequently, to show $\sigma(a)$ is compact it suffices to show $\sigma(a)$ is closed, or equivalently that $\rho(a)$ is open. Let $G \subseteq A$ be the invertible elements, which we know is open. Now $\mathbb{C} \ni z \mapsto (z-a) \in A$ is continuous, and $\rho(a)$ is the inverse image of G, hence $\rho(a)$ is open.

Now, denote $F(z) := (z-a)^{-1}$ for $z \in \rho(a)$. Fix $z_0 \in \rho(a)$ and let $h \in \mathbb{C} \setminus \{0\}$ be small enough so that $z_0 + h \in \rho(a)$. Then

$$\frac{1}{h}[F(z_0+h) - F(z_0)] = \frac{1}{h}(z_0+h-a)^{-1}[1 - (z_0+h-a)\cdot(z_0-a)^{-1}]$$

$$= \frac{1}{h}(z_0+h-a)^{-1}[(z_0-a) - (z_0+h-a)](z_0-a)^{-1}$$

$$= \frac{1}{h}(z_0+h-a)^{-1}[-h](z_0-a)^{-1}$$

$$= -(z_0+h-a)^{-1}(z_0-a)^{-1} \xrightarrow{h \to 0} -(z_0-a)^{-2}$$

where we have used that $G \ni b \mapsto b^{-1} \in G$ is continuous. So $F'(z) = -(z-a)^{-2}$, which is continuous, so F is analytic on $\rho(a)$.

It remains to show $\sigma(a) \neq \phi$. Suppose not, then $\rho(a) = \mathbb{C}$ and so F is an entire function. But

$$\|F(z)\| = \|(z-a)^{-1}\| = \frac{1}{|z|}\|(1-a/z)\| \xrightarrow{z\to\infty} 0 \cdot 1 = 0,$$

so F is also bounded. Louiville's Theorem then implies F is constant. This is a contradiction since, for example, $F' \neq 0$. Thus $\rho(a) \neq \mathbb{C}$ and $\sigma(a) \neq \phi$. $\square$

Def Let $\mathcal{A}$ be a unital Banach algebra. The spectral radius of $a \in \mathcal{A}$ is the quantity

$$r(a) := \sup_{z \in \sigma(a)} |z|$$

- The previous theorem implies $0 \leq r(a) < \infty$ and moreover $\exists\, z \in \sigma(a)$ with $r(a) = |z|$.

Ex ① Let X be a compact Hausdorff space. For $f \in C(X)$ we have seen $\sigma(f) = f(X)$. Hence $r(f) = \|f\|_\infty$.

② For $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in M_2(\mathbb{C})$, $\sigma(A) = \{0\}$ so $r(A) = 0 < \|A\| = 1$. Note that $A^2 = 0$, so $r(A) = \|A^2\|$.

Prop Let $\mathcal{A}$ be a unital Banach algebra. For all $a \in \mathcal{A}$

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof Fix $a \in \mathcal{A}$ and let $\mathcal{U} = \{z \in \mathbb{C} : z = 0 \text{ or } z^{-1} \in \rho(a)\}$. Define $f : \mathcal{U} \to \mathcal{A}$ by

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ (z^{-1} - a)^{-1} & \text{otherwise} \end{cases}$$

Then f is analytic (Exercise). Using the lemma, we see that for $|z| < \|a\|^{-1}$

$$f(z) = z(1 - za)^{-1} = z \sum_{n=0}^{\infty} a^n z^n$$

By complex analysis (namely uniqueness of analytic extensions), we know this power series is valid on any $B(0,R) \subseteq \mathcal{U}$. In particular, for

$$R := \operatorname{dist}(0, \partial\mathcal{U}) = \operatorname{dist}(0, \sigma(a)^{-1}) = \inf\{|z| : z^{-1} \in \sigma(a)\} = r(a)^{-1}.$$

On the other hand, by direct inspection of the power series we have

$$r(a) = R^{-1} = \limsup_{n\to\infty} \|a^n\|^{1/n}.$$

Conversely, if $z \in \sigma(a)$, then $z^n \in \sigma(a^n)$ since $a^n - z^n = (a-z)(a^{n-1} + a^{n-2}z + \cdots + z^{n-1})$. Thus the previous theorem implies $|z^n| \leq \|a^n\|$ or $|z| \leq \|a^n\|^{1/n}$. Consequently, $|z| \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$ for all $z \in \sigma(a)$. This implies $r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n} \leq \limsup_{n\to\infty} \|a^n\|^{1/n} = r(a)$. $\square$